4 Local and trivial symmetries, Noethers theorems

4.1 Definition of infinitesimal local transformations and symmetries

Local symmetries, also called gauge symmetries[10], are generated by transformations (gauge transformations) containing functions which can depend *arbitrarily* on the coordinates x^μ of the base space. We shall denote such functions by the letter λ and infinitesimal gauge transformations by $\delta_\lambda \phi^i$,

$$\delta_\lambda \phi^i = R^i \lambda, \quad R^i = r^i([\phi], x) + r^{i\mu}([\phi], x)\, \partial_\mu + \dots \\ = \sum_k r^{i\mu_1 \dots \mu_k}([\phi], x)\, \partial_{\mu_1} \dots \partial_{\mu_k} \; . \tag{4.1}$$

This means that gauge transformations may involve λ and its derivatives in addition to the fields, their derivatives and the coordinates of the base space. For later purposes we have written such transformations in terms of operators R^i which may depend on the fields and their derivatives and on the base space coordinates. Analogously to our definition of infinitesimal global symmetry transformations, $\delta_\lambda \phi^i$ represents the difference $\tilde{\phi}^i(x) - \phi^i(x)$ linearly in λ, where $\tilde{\phi}^i(x)$ is a finite gauge transformation taken at the same arguments x^μ as $\phi^i(x)$. Accordingly, δ_λ vanishes on all x^μ and commutes with all ∂_μ,

$$\delta_\lambda x^\mu = 0, \quad [\delta_\lambda, \partial_\mu] = 0, \tag{4.2}$$

and has the jet space representation

$$\delta_\lambda = (R^i \lambda) \frac{\partial}{\partial \phi^i} + \partial_\mu (R^i \lambda) \frac{\partial}{\partial \phi^i_{,\mu}} + \dots \tag{4.3}$$

Finite gauge transformations arise from infinitesimal ones according to

$$\tilde{\phi}^i(x) = \Big(\exp(\delta_\lambda) \phi^i \Big)(x). \tag{4.4}$$

The definition of local symmetries of action functionals is analogous to the definition of global symmetries.

Definition: δ_λ is called an infinitesimal local symmetry (gauge symmetry) of an action $S[\phi] = \int d^n x\, L([\phi], x)$ if

$$\delta_\lambda L([\phi], x) = \partial_\mu K^\mu([\phi, \lambda], x) \tag{4.5}$$

for some functions $K^\mu([\phi, \lambda], x)$.

4.2 Example: spinor field coupled to electromagnetic field

In sections 3.4.2 and 3.4.3 we have constructed Poincaré invariant actions $S[A]$ and $S[\psi]$ for vector fields A_μ and spinor fields ψ, respectively. $S[A]$ describes, in particular, free electromagnetic fields (A_μ are the electromagnetic potentials; the Euler-Lagrange equations deriving from $S[A]$ are the inhomogeneous Maxwell equations in the vacuum). $S[\psi]$ describes a free spinor spinor field (such as the free electron field; the Euler-Lagrange equations deriving from $S[\psi]$ are the Dirac equations). Electromagnetic interactions of a spinor field (e.g. electron field)

[10] We shall use the terms local symmetry and gauge symmetry synonymously. The terminology used in the literature varies, however. Some authors reserve the term gauge symmetry for particular local symmetries of the Yang-Mills type.

are described by a gauge invariant action $S[A,\psi]=\int d^nx\,L([A,\psi])$ that contains $S[A]+S[\psi]$ and an additional interaction term:

$$L([A,\psi]) = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\,\bar\psi\Gamma^\mu\mathcal{D}_\mu\psi + \mathrm{i}\,m\bar\psi\psi, \tag{4.6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{(electromagnetic field strengths)}, \tag{4.7}$$

$$\mathcal{D}_\mu\psi = \partial_\mu\psi - \mathrm{i}eA_\mu\psi \quad \text{(covariant derivatives of } \psi). \tag{4.8}$$

Here m and e denote the mass and electric charge of the particles associated to ψ (e.g. mass and electric charge of the electron). The action is invariant under the infinitesimal gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu\lambda, \quad \delta_\lambda\psi = \mathrm{i}e\lambda\psi, \quad \delta_\lambda\bar\psi = -\mathrm{i}e\lambda\bar\psi, \tag{4.9}$$

where $\lambda(x)$ is an arbitrary real function ($\delta_\lambda\bar\psi = -\mathrm{i}e\lambda\bar\psi$ follows from $\delta_\lambda\psi = \mathrm{i}e\lambda\psi$ according to $\delta_\lambda\bar\psi = (\delta_\lambda\psi)^\dagger\Gamma_0 = (\mathrm{i}e\lambda\psi)^\dagger\Gamma_0 = -\mathrm{i}e\lambda\bar\psi$). This implies the following gauge transformations of $F_{\mu\nu}$ and $\mathcal{D}_\mu\psi$:

$$\delta_\lambda F_{\mu\nu} = \partial_\mu(\delta_\lambda A_\nu) - \partial_\nu(\delta_\lambda A_\mu) = \partial_\mu\partial_\nu\lambda - \partial_\nu\partial_\mu\lambda = 0 \tag{4.10}$$

$$\begin{aligned}
\delta_\lambda\mathcal{D}_\mu\psi &= \partial_\mu(\delta_\lambda\psi) - \mathrm{i}e(\delta_\lambda A_\mu)\psi - \mathrm{i}eA_\mu(\delta_\lambda\psi)\\
&= \partial_\mu(\mathrm{i}e\lambda\psi) - \mathrm{i}e(\partial_\mu\lambda)\psi - \mathrm{i}eA_\mu(\mathrm{i}e\lambda\psi)\\
&= \mathrm{i}e(\partial_\mu\lambda)\psi + \mathrm{i}e\lambda\partial_\mu\psi - \mathrm{i}e(\partial_\mu\lambda)\psi - \mathrm{i}eA_\mu(\mathrm{i}e\lambda\psi)\\
&= \mathrm{i}e\lambda\partial_\mu\psi - \mathrm{i}eA_\mu(\mathrm{i}e\lambda\psi)\\
&= \mathrm{i}e\lambda(\partial_\mu\psi - \mathrm{i}eA_\mu\psi) = \mathrm{i}e\lambda\mathcal{D}_\mu\psi.
\end{aligned} \tag{4.11}$$

Notice that $\mathcal{D}_\mu\psi$ transforms in the same way as ψ simply by multiplication with $\mathrm{i}e\lambda$ whereas $\partial_\mu\psi$ transforms according to $\delta_\lambda\partial_\mu\psi = \mathrm{i}e(\partial_\mu\lambda)\psi + \mathrm{i}e\lambda\partial_\mu\psi$. The term $\mathrm{i}e(\partial_\mu\lambda)\psi$ drops out in $\mathcal{D}_\mu\psi$ because it is compensated for by the term $-\mathrm{i}e(\delta_\lambda A_\mu)\psi$. Hence, the term $-\mathrm{i}eA_\mu\psi$ in $\mathcal{D}_\mu\psi$ arranges for $\delta_\lambda\mathcal{D}_\mu\psi$ not to contain a derivative of λ and thus to transform in a simple way. It is now straightforward to verify the gauge invariance of the Lagrangian:

$$\delta_\lambda(F_{\mu\nu}F^{\mu\nu}) = 0 \tag{4.12}$$

$$\delta_\lambda(\bar\psi\Gamma^\mu\mathcal{D}_\mu\psi) = (\delta_\lambda\bar\psi)\Gamma^\mu\mathcal{D}_\mu\psi + \bar\psi\Gamma^\mu\delta_\lambda\mathcal{D}_\mu\psi = (-\mathrm{i}e\lambda\bar\psi)\Gamma^\mu\mathcal{D}_\mu\psi + \bar\psi\Gamma^\mu(\mathrm{i}e\lambda\mathcal{D}_\mu\psi) = 0 \tag{4.13}$$

$$\delta_\lambda(\bar\psi\psi) = (\delta_\lambda\bar\psi)\psi + \bar\psi(\delta_\lambda\psi) = (-\mathrm{i}e\lambda\bar\psi)\psi + \bar\psi(\mathrm{i}e\lambda\psi) = 0 \tag{4.14}$$

$$\Rightarrow \quad \delta_\lambda L([A,\psi]) = 0. \tag{4.15}$$

4.3 Comments

- The example in section 4.2 illustrates typical features of many gauge theories. The electromagnetic potentials A_μ are examples of so-called ‘gauge fields’. Gauge fields typically contain in their gauge transformations terms with derivatives of functions λ and enable one to construct covariant derivatives such as $\mathcal{D}_\mu\psi$ in the above example. Covariant derivatives of fields extend partial derivatives and have simpler gauge transformations than the latter; typically these gauge transformations do not contain derivatives of the functions λ as in the above example (cf. (4.11)), or they contain derivatives of these functions only in a particular way. The electromagnetic field strengths $F_{\mu\nu}$ are (rather simple) examples of ‘field strengths’ or ‘curvature fields’ constructed out of gauge fields. The latter are also characterized by simple gauge transformations (in the electromagnetic case one even has $\delta_\lambda F_{\mu\nu} = 0$, cf. (4.10)). Usually they are related to the covariant derivatives because they occur in the commutators of covariant derivatives, and they often have a geometric meaning or interpretation (a famous example is the Riemann tensor which describes geometric properties of Riemannian manifolds and plays a central role in general relativity).

- The requirement of gauge invariance controls and restricts, in particular, the interactions of fields. These interactions are represented in the Lagrangian by terms of third or higher order in the fields. E.g., the only interaction terms in (4.6) are $e\bar{\psi}\Gamma^\mu A_\mu\psi$ which describe the coupling of the spinor field to the electromagnetic field. Gauge invariance rules out many interaction terms that would be permitted by Poincaré symmetry alone, such as $A_\mu A^\mu A_\nu A^\nu$ in the above example.

4.4 Useful mathematical concepts and results

4.4.1 Equivalence of functions and functionals

It is often useful, especially in the context of symmetries, to call two functions $f([\phi],x)$ and $g([\phi],x)$ equivalent (with equivalence denoted by $\simeq$) if they differ by a total divergence:

$$f([\phi],x)\simeq g([\phi],x)\quad :\Leftrightarrow\quad \exists\, k^\mu([\phi],x):\quad f([\phi],x)-g([\phi],x)=\partial_\mu k^\mu([\phi],x). \tag{4.16}$$

In particular $f([\phi],x)\simeq 0$ means $f([\phi],x)=\partial_\mu k^\mu([\phi],x)$. Functionals $\int d^nx\, f([\phi],x)$ and $\int d^nx\, g([\phi],x)$ are called equivalent if $f([\phi],x)\simeq g([\phi],x)$.

4.4.2 'Variational formula'

Let $\delta\phi^i$ denote any variation or transformation of the fields which is 'prolongated' to derivatives of the fields according to $[\partial_\mu,\delta]=0$, $\delta x^\mu=0$, and which is a derivation on functions of the $\phi^i,\phi^i_{,\mu},\phi^i_{,\mu\nu},\dots,x^\mu$. This means $\delta\phi^i_{,\mu}=\partial_\mu\delta\phi$, $\delta\phi^i_{,\mu\nu}=\partial_\nu\partial_\mu\delta\phi^i$, …, and

$$\delta=(\delta\phi^i)\,\frac{\partial}{\partial\phi^i}+(\partial_\mu\delta\phi^i)\,\frac{\partial}{\partial\phi^i_{,\mu}}+(\partial_\nu\partial_\mu\delta\phi^i)\,\frac{\partial^S}{\partial\phi^i_{,\mu\nu}}+\dots \tag{4.17}$$

(see section 2.6 for the definition of $\dfrac{\partial^S}{\partial\phi^i_{,\mu\nu}}$). These δ fulfill the 'variational formula'

$$\forall f([\phi],x):\quad \delta f([\phi],x)\simeq(\delta\phi^i)\,\frac{\delta f([\phi],x)}{\delta\phi^i} \tag{4.18}$$

with $\simeq$ as defined in (4.16). (4.18) holds because of

$$\begin{aligned}\delta f&=(\delta\phi^i)\,\frac{\partial f}{\partial\phi^i}+(\partial_\mu\delta\phi^i)\frac{\partial f}{\partial\phi^i_{,\mu}}+\dots\\&=(\delta\phi^i)\,\frac{\partial f}{\partial\phi^i}+\partial_\mu\Big((\delta\phi^i)\,\frac{\partial f}{\partial\phi^i_{,\mu}}\Big)-(\delta\phi^i)\,\partial_\mu\frac{\partial f}{\partial\phi^i_{,\mu}}+\dots\\&=(\delta\phi^i)\,\frac{\delta f}{\delta\phi^i}+\partial_\mu\Big((\delta\phi^i)\,\frac{\partial f}{\partial\phi^i_{,\mu}}+\dots\Big).\end{aligned}$$

4.4.3 Adjoint operator

Let P be an operator defined on functions $f=f([\phi],x)$ according to

$$\forall f:\ Pf=\sum_{k\geq 0}p^{\mu_1\cdots\mu_k}\partial_{\mu_1}\dots\partial_{\mu_k}f=pf+p^\mu\partial_\mu f+\dots \tag{4.19}$$

with some coefficient functions $p^{\mu_1\cdots\mu_k} = p^{\mu_1\cdots\mu_k}([\phi], x)$. We define an operator $P^\dagger$ and call it the operator adjoint to P according to:

$$\forall f: \ P^\dagger f := \sum_{k\geq 0}(-)^k \partial_{\mu_1}\dots\partial_{\mu_k}(f\, p^{\mu_1\cdots\mu_k}) = pf - \partial_\mu(f\, p^\mu) + \dots \tag{4.20}$$

This definition implies, with $\simeq$ as in (4.16),

$$\forall f, g: \quad f\,(Pg) \simeq (P^\dagger f)\, g \tag{4.21}$$

which motivates the notation $P^\dagger$. (4.21) holds because of

$$\begin{aligned} f\,(Pg) &= f(pg + p^\mu\partial_\mu g + \dots) = fpg + \partial_\mu(fp^\mu g) - \partial_\mu(fp^\mu)g + \dots \\ &= (P^\dagger f)g + \partial_\mu(fp^\mu g + \dots) \simeq (P^\dagger f)g. \end{aligned}$$

4.4.4 Algebraic Poincaré lemma

a) A function $f([\phi], x)$ has vanishing Euler-Lagrange derivatives w.r.t. all fields ϕ^i if and only if $f([\phi], x)$ is a total divergence,

$$\frac{\delta}{\delta\phi^i} f([\phi], x) = 0 \quad \forall\, \phi^i \quad \Leftrightarrow \quad \exists\, J^\mu([\phi], x):\ f([\phi], x) = \partial_\mu J^\mu([\phi], x). \tag{4.22}$$

b) Consider a set of functions $J^{\mu_1\cdots\mu_k}([\phi], x)$ related by the permutation symmetry

$$J^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_k} = -J^{\mu_1\cdots\mu_j\cdots\mu_i\cdots\mu_k} \quad \forall i, j\ (i \neq j). \tag{4.23}$$

One has in n dimensions:

$$0 < k < n: \quad \partial_{\mu_k} J^{\mu_1\cdots\mu_k} = 0 \quad \Leftrightarrow \quad \exists\, J^{\mu_1\cdots\mu_{k+1}}:\ J^{\mu_1\cdots\mu_k} = \partial_{\mu_{k+1}} J^{\mu_1\cdots\mu_{k+1}}, \tag{4.24}$$

where the functions $J^{\mu_1\cdots\mu_{k+1}}$ fulfill (4.23) too (with $k+1$ in place of k).

c) A function $f([\phi], x)$ which is annihilated by all ∂_μ is a constant function (i.e., it does not depend on the fields, their derivatives or the x^μ at all):

$$\partial_\mu f([\phi], x) = 0 \quad \forall\mu \quad \Leftrightarrow \quad f([\phi], x) = constant. \tag{4.25}$$

Comment: These results can be expressed more compactly in terms of differential forms ω^p and the exterior derivative operator d defined according to

$$\begin{aligned} \omega^p &= \frac{1}{p!(n-p)!}\, dx^{\mu_1}\dots dx^{\mu_p}\epsilon_{\mu_1\dots\mu_n} J^{\mu_{p+1}\cdots\mu_n}, \\ d &= dx^\mu\partial_\mu, \quad dx^\mu dx^\nu = -dx^\nu dx^\mu \quad \forall \mu, \nu. \end{aligned}$$

Then they read

$$\begin{aligned} p = n: &\quad \exists\, \omega^{n-1}:\ \omega^n = d\omega^{n-1} \ \Leftrightarrow\ \frac{\delta\omega^n}{\delta\phi^i} = 0 \quad \forall\phi^i \\ 0 < p < n: &\quad d\omega^p = 0 \ \Leftrightarrow\ \exists\, \omega^{p-1}:\ \omega^p = d\omega^{p-1} \\ p = 0: &\quad d\omega^0 = 0 \ \Leftrightarrow\ \omega^0 = constant. \end{aligned}$$

The case $p = n$ is equivalent to (4.22) (with $\omega^n = (-)^{n-1}d^n x f$), the cases $0 < p < n$ are equivalent to (4.24) for $k = n - p$ (with ω^p as above and $\omega^{p-1} = (-)^{n-1}((p-1)!(n-p+$

$1)!)^{-1}dx^{\mu_1}\dots dx^{\mu_{p-1}}\epsilon_{\mu_1\dots\mu_n}J^{\mu_p\cdots\mu_n})$, and the case $p=0$ is equivalent to (4.25) (with $\omega^0=f$). These results are reminiscent of the so-called Poincaré lemma for ordinary differential forms $(p!)^{-1}dx^{\mu_1}\dots dx^{\mu_p}\omega_{\mu_1\dots\mu_p}(x)$ and are sometimes referred to as the 'algebraic Poincaré lemma' (for differential forms on jet spaces). They contain the ordinary Poincaré lemma as a special case when the differential forms do not depend on the fields at all.[11]

4.4.5 Fermionic and bosonic fields, Grassmann parity and grading

In view of quantum theory it is useful to distinguish between 'bosonic' and 'fermionic' fields already in classical field theory. Two Fermionic fields anticommmute, two bosonic fields commute, a fermionic field and a bosonic field commute. Examples of fermionic fields are spinor fields, examples of bosonic fields are scalar and vector fields (where spinor, scalar and vector refers to the transformation under Lorentz transformations). To describe the commutation relations we assign a 'grading' ('Grassmann parity') $\sigma(\phi^i)$ to each field which is either 0 (for bosonic fields) or 1 (for fermionic fields),

$$\sigma(\phi^i)\in\{0,1\}. \tag{4.26}$$

The grading determines the commutation relations of the fields according to

$$\phi^i\phi^j=(-)^{\sigma(\phi^i)\sigma(\phi^j)}\phi^j\phi^i. \tag{4.27}$$

This implies $\phi^i\phi^j\phi^k = (-)^{\sigma(\phi^i)\sigma(\phi^j)}\phi^j\phi^i\phi^k = (-)^{\sigma(\phi^i)\sigma(\phi^j)}(-)^{\sigma(\phi^i)\sigma(\phi^k)}\phi^j\phi^k\phi^i = (-)^{\sigma(\phi^i)(\sigma(\phi^j)+\sigma(\phi^k))}\phi^j\phi^k\phi^i$, i.e., $\sigma(\phi^j\phi^k)=\sigma(\phi^j)+\sigma(\phi^k)$. Extending this to products of arbitrarily many fields, one concludes that the grading is additive (modulo 2) for products of fields,

$$\sigma(\phi^{i_1}\phi^{i_2}\cdots\phi^{i_r})=\sigma(\phi^{i_1})+\sigma(\phi^{i_2})+\cdots+\sigma(\phi^{i_r})\quad(\mathrm{mod}\ 2). \tag{4.28}$$

Consistency requries that, e.g.,

$$\frac{\partial}{\partial\phi^1}(\phi^1\phi^2)\stackrel{!}{=}(-)^{\sigma(\phi^1)\sigma(\phi^2)}\frac{\partial}{\partial\phi^1}(\phi^2\phi^1).$$

This imposes that the derivative w.r.t. a field has the same grading as the field itself:

$$\sigma(\frac{\partial}{\partial\phi^i})=\sigma(\phi^i),\quad \text{i.e.,}\quad \forall f:\quad \frac{\partial}{\partial\phi^i}(\phi^j f)=\delta_i^j\,f+(-)^{\sigma(\phi^i)\sigma(\phi^j)}\phi^j\frac{\partial}{\partial\phi^i}\,f. \tag{4.29}$$

The base space coordinates are even graded and so are the derivatives ∂_μ,

$$\sigma(x^\mu)=\sigma(\partial_\mu)=0. \tag{4.30}$$

Global or local symmetry transformations do not alter the grading, i.e. they are even graded

$$\sigma(\delta_\varepsilon)=0,\quad \sigma(\delta_\lambda)=0. \tag{4.31}$$

This reflects that finite transformations $\exp(\delta_\varepsilon)\phi$ or $\exp(\delta_\lambda)\phi$ should not alter the grading of a field ϕ.

Furthermore we impose that Lagrangians are bosonic (i.e., each monomial in a Lagrangian contains an even number of fermionic fields),

$$\sigma(L([\phi],x))=0. \tag{4.32}$$

[11] We have disregarded here possible global obstructions to the existence of differential forms, i.e., in general the results hold only locally. Proofs of the algebraic Poincaré lemma can be found e.g. in Ref. [3], or in section 4 of G. Barnich et al, Phys. Rept. 338 (2000) 439 [hep-th/0002245] and in references mentioned there.

4.5 Noether identities and Noethers second theorem

Using the terminology introduced above, Noethers second theorem can be formulated as follows:

$\delta_\lambda \phi^i = R^i\lambda$ generates a local symmetry of an action $S[\phi] = \int d^nx\, L([\phi], x)$ if and only if the Euler-Lagrange derivatives of L fulfill the identity

$$(-)^{\sigma(\phi^i)} R^{i\dagger} \frac{\delta L}{\delta \phi^i} = 0 \tag{4.33}$$

where $\sigma(\phi^i)$ denotes the grading of ϕ^i (see section 4.4.5).

Proof:
'⇒': We assume that δ_λ is a local symmetry of $S = \int d^nx\, L$, i.e., $\delta_\lambda L = \partial_\mu K^\mu$ for all functions $\lambda(x)$. This implies (4.33) by applying the Euler-Lagrange derivative w.r.t. λ to the equation $\delta_\lambda L = \partial_\mu K^\mu$:

$$\delta_\lambda L = \partial_\mu K^\mu \quad \Rightarrow \quad \frac{\delta}{\delta\lambda}\Big(\delta_\lambda L\Big) = \frac{\delta}{\delta\lambda}\Big(\partial_\mu K^\mu\Big) \quad \Rightarrow \quad (-)^{\sigma(\phi^i)} R^{i\dagger}\frac{\delta L}{\delta\phi^i} = 0.$$

Since it might not be evident how the factor $(-)^{\sigma(\phi^i)}$ arises, we shall now comment in some more detail on this derivation. Applying the variational formula (4.18) to $\delta_\lambda L = \partial_\mu K^\mu$, one obtains $(\delta_\lambda \phi^i)\,\frac{\delta L}{\delta\phi^i} \simeq 0$ (with equivalence $\simeq$ as in (4.16)). This implies

$$\begin{aligned} 0 \simeq (\delta_\lambda\phi^i)\,\frac{\delta L}{\delta\phi^i} &= (-)^{\sigma(\phi^i)\sigma(\phi^i)}\,\frac{\delta L}{\delta\phi^i}\,(\delta_\lambda\phi^i) = (-)^{\sigma(\phi^i)}\,\frac{\delta L}{\delta\phi^i}\,(\delta_\lambda\phi^i) \\ &= (-)^{\sigma(\phi^i)}\,\frac{\delta L}{\delta\phi^i}\,(R^i\lambda) \simeq (-)^{\sigma(\phi^i)}\Big(R^{i\dagger}\frac{\delta L}{\delta\phi^i}\Big)\lambda. \end{aligned} \tag{4.34}$$

Here we used that $\sigma(\frac{\delta L}{\delta\phi^i}) = \sigma(\phi^i)$ and $\sigma(\delta_\lambda\phi^i) = \sigma(\phi^i)$, as follows from equations (4.29)–(4.32). Furthermore we used that $(\sigma(\phi^i))^2 = \sigma(\phi^i)$ (this holds owing to $\sigma(\phi^i) \in \{0,1\}$) and applied equation (4.21). The Euler Lagrange derivative of (4.34) w.r.t. λ yields (4.33) because of (4.22) (since λ is an arbitrary function, it plays the same role as the fields in (4.22)).

'⇐': We multiply (4.33) from the right by λ, then use (4.34) reading the latter from the right to the left, and finally use the variational formula (4.18) to conclude $\delta_\lambda L \simeq 0$ which shows that δ_λ is a local symmetry of $S = \int d^nx\, L$.

Example:
Let us consider again the example in section 4.2. In this case it is convenient to choose the set of fields $\{\phi^i\} = \{A_\mu, \psi^\alpha, \bar\psi_\alpha\}$ (for complex fields one may take the real and imaginary parts as independent fields, or any linearly independent (complex) linear combinations thereof; hence, in the case of a spinor field one may take the components of ψ and $\bar\psi$ as a set of fields). Since spinor fields ψ are fermionic, the Noether identities (4.33) deriving from (4.9) read

$$-\partial_\mu\,\frac{\delta L}{\delta A_\mu} - \frac{\delta L}{\delta\psi^\alpha}\,(\mathrm{i}e\psi^\alpha) - \frac{\delta L}{\delta\bar\psi_\alpha}\,(-\mathrm{i}e\bar\psi_\alpha) = 0. \tag{4.35}$$

Let us verify this identity explicitly. One has

$$\frac{\delta L}{\delta A_\mu} = \partial_\nu F^{\nu\mu} + e\bar\psi\Gamma^\mu\psi \tag{4.36}$$

$$\frac{\delta L}{\delta\psi^\alpha} = \mathrm{i}(\partial_\mu\bar\psi\Gamma^\mu)_\alpha - e(\bar\psi\Gamma^\mu)_\alpha A_\mu - \mathrm{i}m\bar\psi_\alpha \tag{4.37}$$

$$\frac{\delta L}{\delta\bar\psi_\alpha} = \mathrm{i}(\Gamma^\mu\partial_\mu\psi)^\alpha + eA_\mu(\Gamma^\mu\psi)^\alpha + \mathrm{i}m\psi^\alpha \tag{4.38}$$

This gives

$$
\begin{aligned}
-\,\partial_\mu \frac{\delta L}{\delta A_\mu} &= -e\partial_\mu(\bar\psi\Gamma^\mu\psi)\\
-\,\frac{\delta L}{\delta\psi^\alpha}\,(\mathrm{i}e\psi^\alpha) &= e(\partial_\mu\bar\psi)\Gamma^\mu\psi + \mathrm{i}e^2\bar\psi\Gamma^\mu\psi A_\mu - em\bar\psi\psi\\
-\,\frac{\delta L}{\delta\bar\psi_\alpha}\,(-\mathrm{i}e\bar\psi_\alpha) &= e\bar\psi\Gamma^\mu\partial_\mu\psi - \mathrm{i}e^2 A_\mu\bar\psi\Gamma^\mu\psi + em\bar\psi\psi
\end{aligned}
$$

where we used $\partial_\mu\partial_\nu F^{\nu\mu} = 0$ (this follows from $F^{\nu\mu} = -F^{\mu\nu}$ and $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$) and the fact that ψ and $\bar\psi$ are fermionic. (4.35) is now obvious.

4.6 Trivial local symmetries

Example
Consider the transformations

$$
\delta_\lambda\phi^i = \lambda\, m^{ji}([\phi],x)\,\frac{\delta L}{\delta\phi^j}\ ,\quad m^{ji}([\phi],x) = -m^{ij}([\phi],x) \tag{4.39}
$$

where L is a Lagrangian involving at least two different fields ($i = 1,\dots,N$ with $N > 1$), and all the fields are bosonic. The transformations (4.39) generate a local symmetry of $S = \int d^n x L$ according to our definition because of

$$
\delta_\lambda L \simeq (\delta_\lambda\phi^i)\frac{\delta L}{\delta\phi^i} = \lambda\, m^{ji}([\phi],x)\,\frac{\delta L}{\delta\phi^j}\,\frac{\delta L}{\delta\phi^i} = 0
$$

where we used the variational formula (4.18), and the last equality (= 0) holds because of the antisymmetry of m^{ji}. Notice that this argument applies to every Lagrangian with bosonic fields and every set of functions $m^{ji}([\phi],x)$ with $m^{ji} = -m^{ij}$.

General case
Consider the following transformations

$$
\delta_\lambda\phi^i = \sum_{k,m\geq 0}(-)^k\partial_{\mu_1}\dots\partial_{\mu_k}\Big(M^{j(\nu_1\dots\nu_m)i(\mu_1\dots\mu_k)}([\phi,\lambda],x)\partial_{\nu_1}\dots\partial_{\nu_m}\frac{\delta L}{\delta\phi^j}\Big), \tag{4.40}
$$

where

$$
M^{j(\nu_1\dots\nu_m)i(\mu_1\dots\mu_k)}([\phi,\lambda],x) = -(-)^{\sigma(\phi^i)\sigma(\phi^j)}M^{i(\mu_1\dots\mu_k)j(\nu_1\dots\nu_m)}([\phi,\lambda],x). \tag{4.41}
$$

Transformations (4.40) are not restricted to Lagrangians containing only bosonic fields and exist even when the Lagrangian contains only one field (see exercise 15). (4.41) implies that all transformations (4.40) generate gauge symmetries according to our definition,

$$
\begin{aligned}
\delta_\lambda L &\simeq (\delta_\lambda\phi^i)\frac{\delta L}{\delta\phi^i}\\
&\simeq \sum_{k,m} M^{j(\nu_1\dots\nu_m)i(\mu_1\dots\mu_k)}([\phi,\lambda],x)\Big(\partial_{\nu_1}\dots\partial_{\nu_m}\frac{\delta L}{\delta\phi^j}\Big)\Big(\partial_{\mu_1}\dots\partial_{\mu_k}\frac{\delta L}{\delta\phi^i}\Big) = 0.
\end{aligned}
$$

Since the functions $M^{j(\nu_1\dots\nu_m)i(\mu_1\dots\mu_k)}$ are completely arbitrary except for the permutation properties (4.41), every action has infinitely many local symmetries of this type. For obvious reasons, these gauge symmetries are called trivial gauge symmetries. Notice in particular that the gauge transformations of these symmetries vanish weakly ($\delta_\lambda\phi^i \approx 0$ with weak equality $\approx$

as in section 2.2)[12]. Accordingly, the operators R^i and $R^{i\dagger}$ of trivial gauge symmetries vanish weakly too.

A sufficient criterion for a symmetry to be nontrivial is thus that the gauge transformations or, equivalently, the operators R^i do not all vanish weakly.

4.7 Trivial and equivalent global symmetries

In section 4.6 it was shown that every action has infinitely many local symmetries because of the presence of trivial local symmetries. A local symmetry δ_λ involves an arbitrary function which we denoted generically by λ. Since δ_λ is a symmetry for every choice of $\lambda(x)$, it is also symmetry when we replace $\lambda(x)$ by some function $f([\phi],x)\varepsilon$. In this manner δ_λ induces a global symmetry of the action generated by the transformations $\delta_\varepsilon\phi^i = Q^i([\phi],x)\varepsilon$ with $Q^i = R^i f([\phi],x)$, where the R^i are the operators associated with δ_λ (cf. (4.1)). Since $f([\phi],x)$ is completely arbitrary (except that $f\varepsilon$ should have the same grading as λ) and since every action has infinitely many local symmetries, every action has infinitely many global symmetries too.

Global symmetries which arise in this manner from local symmetries are called trivial global symmetries. Two global symmetries which differ only by a trivial global symmetry are called equivalent. More precisely, two global symmetries are called equivalent if the respective functions $Q^i([\phi],x)$ differ only by $R^i f([\phi],x)$ for some function $f([\phi],x)$ and a set of operators R^i generating a local symmetry (whether or not that local symmetry is trivial). Denoting this equivalence by $\sim$ and using Noethers second theorem (cf. section 4.5), we summarize this definition as follows:

$$\delta_\varepsilon \sim \delta'_{\varepsilon'} \quad :\Leftrightarrow \quad \exists f, R^i: \quad Q^i - Q^{i\prime} = R^i f, \quad (-)^{\sigma(\phi^i)} R^{i\dagger}\frac{\delta L}{\delta\phi^i} = 0. \tag{4.42}$$

4.8 Trivial and equivalent conservation laws

In section 2.2 we defined a conserved current as a set of functions $J^\mu([\phi],x)$ satisfying $\partial_\mu J^\mu([\phi],x) \approx 0$. If $n > 1$, $\partial_\mu J^\mu([\phi],x)$ is trivially fulfilled if $J^\mu \approx \partial_\nu K^{\mu\nu}$ for some functions $K^{\nu\mu}([\phi],x)$ with $K^{\mu\nu} = -K^{\nu\mu}$ because $\partial_\mu J^\mu \approx \partial_\mu\partial_\nu K^{\mu\nu} = 0$ (the latter holds because of $K^{\mu\nu} = -K^{\nu\mu}$ and $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$). This motivates us to call a conserved current trivial if $J^\mu \approx \partial_\nu K^{\nu\mu}$, and to call two currents equivalent ($J^\mu \sim J^{\mu\prime}$) if they differ only by a trivial current,

$$n > 1: \quad J^\mu \sim J^{\mu\prime} \quad :\Leftrightarrow \quad \exists\, K^{\mu\nu}: \quad J^\mu - J^{\mu\prime} \approx \partial_\nu K^{\mu\nu}, \quad K^{\nu\mu} = -K^{\mu\nu} \tag{4.43}$$

where we omitted the arguments $[\phi],x$ of the J's and K's for notational convenience.

If $n = 1$, a conserved current is just a conserved function (a constant of motion) and the $K^{\mu\nu}$ are replaced by constant functions:

$$n = 1: \quad J \sim J' \quad :\Leftrightarrow \quad J - J' \approx \textit{constant}. \tag{4.44}$$

[12]Under fairly general assumptions on the Lagrangian one can prove that a gauge symmetry whose gauge transformations vanish weakly necessarily takes the form (4.40), see e.g. appendix 6.A of G. Barnich et al, Phys. Rept. 338 (2000) 439 [hep-th/0002245].

4.9 Noethers first theorem (extended version)

We have already mentioned in sections 1.8 and 2.4 that a bijective correspondence of global symmetries and conserved currents (resp. constants of motion, if $n = 1$) only holds for equivalence classes of global symmetries and conserved currents. The respective equivalences are those described in sections 4.7 and 4.8. Armed with these definitions of equivalent global symmetries and conserved currents, one can prove under fairly general assumptions[13]:

The equivalence classes of global symmetries and the equivalence classes of conserved currents (if $n > 1$), resp. the equivalence classes of constants of motion (if $n = 1$) correspond one-to-one. That is, for each equivalence class of global symmetries there is exactly one equivalence class of conserved currents (if $n > 1$), resp. of constants of motion (if $n = 1$). In particular, trivial global symmetries correspond to trivial conserved currents (if $n > 1$), resp. to trivial constants of motion (if $n = 1$).

We cannot provide here the complete proof of this theorem as this would require additional mathematical tools and results. However we can prove at least part of it. Namely the theorem as stated above includes the result that conserved currents arising from gauge symmetries through $\lambda = f([\phi], x)\varepsilon$ (with $\sigma(\lambda) = \sigma(f\varepsilon)$) are always trivial in the sense of section 4.8. Let us denote a transformation of this type by δ_f,

$$\delta_f \phi^i = R^i f([\phi], x)\varepsilon. \tag{4.45}$$

Since δ_λ is a gauge symmetry (by assumption), we have $\delta_f L = \partial_\mu K_f^\mu$ for some K_f^μ. Using the variational formula (4.18), we conclude

$$(\delta_f \phi^i)\,\frac{\delta L}{\delta \phi^i} = \partial_\mu J_f^\mu \tag{4.46}$$

for some J_f^μ. Hence, J_f^μ is a conserved current (according to (4.46) one has $\partial_\mu J_f^\mu \approx 0$). To show that this current is trivial, we multiply the Noether identity (4.33) from the right by $f\varepsilon$,

$$(-)^{\sigma(\phi^i)}\Big(R^{i\dagger}\,\frac{\delta L}{\delta \phi^i}\Big) f\varepsilon = 0. \tag{4.47}$$

Applying now (4.21) and using $\sigma(\lambda) = \sigma(f\varepsilon)$ we obtain

$$0 = (-)^{\sigma(\phi^i)}\,\frac{\delta L}{\delta \phi^i}\,R^i f\varepsilon - \partial_\mu\Big(S_f^{i\mu}\,\frac{\delta L}{\delta \phi^i}\Big) = (R^i f\varepsilon)\,\frac{\delta L}{\delta \phi^i} - \partial_\mu\Big(S_f^{i\mu}\,\frac{\delta L}{\delta \phi^i}\Big) \tag{4.48}$$

where $S_f^{i\mu}$ is an operator which acts on $\frac{\delta L}{\delta \phi^i}$ and arises when applying (4.21). Explicitly one has

$$\begin{aligned}(-)^{\sigma(\phi^i)}\Big(R^{i\dagger}\,\frac{\delta L}{\delta \phi^i}\Big) f\varepsilon - (-)^{\sigma(\phi^i)}\,\frac{\delta L}{\delta \phi^i}\,R^i f\varepsilon &= -\partial_\mu\Big((-)^{\sigma(\phi^i)}\,\frac{\delta L}{\delta \phi^i}\,r^{i\mu} f\varepsilon + \dots\Big)\\ &= -\partial_\mu\Big(\underbrace{r^{i\mu} f\varepsilon\,\frac{\delta L}{\delta \phi^i} + \dots}_{S_f^{i\mu}}\Big)\end{aligned}$$

where $r^{i\mu}([\phi], x)$ are the functions occurring in (4.1) and the ellipses denote possible further terms containing functions $r^{i\mu_1\cdots\mu_k}([\phi], x)$ from (4.1) with $k > 1$ (such terms are present only when the gauge transformations $\delta_\lambda \phi^i$ contain second or higher order derivatives of λ). (4.48) gives

$$(\delta_f \phi^i)\,\frac{\delta L}{\delta \phi^i} = \partial_\mu\Big(S_f^{i\mu}\,\frac{\delta L}{\delta \phi^i}\Big). \tag{4.49}$$

[13] G. Barnich et al, Commun. Math. Phys. 174 (1995) 57 [hep-th/9405109].

Equations (4.46) and (4.49) yield

$$\partial_\mu\Big(J^\mu_f - S^{i\mu}_f \frac{\delta L}{\delta\phi^i}\Big) = 0. \qquad (4.50)$$

Using now (4.24) for $k = 1$ we conclude from (4.50)

$$J^\mu_f - S^{i\mu}_f \frac{\delta L}{\delta\phi^i} = \partial_\nu K^{\mu\nu}, \quad K^{\mu\nu} = -K^{\nu\mu} \qquad (4.51)$$

for some functions $K^{\mu\nu}([\phi], x)$. This shows that the current in (4.46) indeed is trivial, for any choice of the function f:

$$J^\mu_f \approx \partial_\nu K^{\mu\nu}, \quad K^{\mu\nu} = -K^{\nu\mu}. \qquad (4.52)$$

4.10 Basis and algebra of global symmetries

We call a set $\{\delta_A\}$ of transformations a basis of the infinitesimal global symmetries of an action $S = \int d^n x\, L$ if every global symmetry transformation δ_ε can be written as a linear combination $\varepsilon k^A \delta_A$ of the δ_A (with constant coefficients k^A) up to a trivial global symmetry (completeness of the basis), and if no nonvanishing linear combination of the δ_A is a trivial global symmetry (independence of the elements of the basis). Notice that δ_A denotes the symmetry transformations without parameters ε^A. Using the notation $\delta_A\phi^i = Q^i_A([\phi], x)$, a basis δ_A can be characterized as follows:

$$Q^i([\phi], x)\,\frac{\delta L}{\delta\phi^i} \simeq 0 \quad \Leftrightarrow \quad Q^i([\phi], x) \sim k^A Q^i_A([\phi], x), \quad k^A = constant, \qquad (4.53)$$

$$k^A Q^i_A([\phi], x) \sim 0 \quad \Leftrightarrow \quad k^A = 0 \quad \forall A. \qquad (4.54)$$

In (4.54) we used that $\delta_\varepsilon L = \partial_\mu K^\mu$ with $\delta_\varepsilon\phi^i = \varepsilon Q^i$ is equivalent to $Q^i \frac{\delta L}{\delta\phi^i} \simeq 0$ by the variational formula (4.18). We note that $\varepsilon^A\delta_A$ are bosonic transformations according to (4.31). Nevertheless δ_A can be a fermionic transformation (then ε^A is a fermionic parameter).

The commutator of two global symmetry transformations is again a global symmetry transformation. This follows on the hand from $\delta_\varepsilon L \simeq 0 \;\Rightarrow\; \delta'_{\varepsilon'}\delta_\varepsilon L \simeq 0$ (which is an immediate consequence of $[\delta'_{\varepsilon'}, \partial_\mu] = 0$) and on the other hand from the fact the commutator of two derivations is again a derivation (this follows from the definition of derivations). For a basis δ_A this means that the graded commutator of two elements of the basis (commutator or anticommutator, depending on the grading of these elements) is equivalent to a linear combination of the δ_A again,

$$[\delta_A, \delta_B\} \sim f_{AB}{}^C \delta_C \qquad (4.55)$$

$$[\delta_A, \delta_B\} := \delta_A\delta_B - (-)^{\sigma(\delta_A)\sigma(\delta_B)}\delta_B\delta_A \qquad (4.56)$$

where $f_{AB}{}^C$ are constant coefficients. According to (4.55), the graded commutators of the δ_A form a graded Lie algebra modulo trivial global symmetry transformations.

4.11 Exercises 14 and 15

Exercise 14: Relativistic point particle
The Lagrangian for a relativistic point particle that is invariant under arbitrary (regular) reparametrizations $t \mapsto \tilde{t}(t)$ of the world line $x^\mu(t)$ is

$$L([x]) = \sqrt{\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}} = \sqrt{\dot{x}^\mu \dot{x}_\mu}.$$

a) Verify that δ_λ with $\delta_\lambda x^\mu = \lambda\,\dot{x}^\mu$ is a local symmetry of the action $S[x] = \int dt\, L([x])$ by showing that, for an arbitrary function $\lambda(t)$:

$$\delta_\lambda L = \frac{d}{dt}\Big(\lambda L\Big)$$

b) Determine and verify explicitly the corresponding Noether identity.

c) Verify that the 'Noether charge' $J = \lambda L - (\delta_\lambda x^\mu)\dfrac{\partial L}{\partial \dot{x}^\mu}$ vanishes identically.

Attention: Here t is the parameter of the world line $x^\mu(t)$ and must not be confused with x^0.

Exercise 15: A trivial local symmetry
Consider the following 1-dimensional example ($n = 1$) with only one dynamical variable x:

$$L([x]) = \frac{m}{2}\,\dot{x}^2, \quad \delta_\lambda\, x = 2\lambda\,\dddot{x} + \dot{\lambda}\ddot{x}$$

Verify that δ_λ is a local symmetry of $S[x] = \int dt\, L([x])$ and that it is trivial according to the terminology introduced above.

Solution of exercise 14
a) One has $\delta_\lambda \dot{x}^\mu = \frac{d}{dt}(\lambda\,\dot{x}^\mu) = \dot{\lambda}\,\dot{x}^\mu + \lambda\,\ddot{x}^\mu$. This gives

$$\begin{aligned}\delta_\lambda L &= \frac{\dot{x}_\mu(\delta_\lambda \dot{x}^\mu)}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = \frac{\dot{x}_\mu(\dot{\lambda}\,\dot{x}^\mu + \lambda\,\ddot{x}^\mu)}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = \frac{\dot{\lambda}\,\dot{x}_\mu \dot{x}^\mu + \lambda\,\dot{x}_\mu \ddot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}\\ &= \dot{\lambda}\,\sqrt{\dot{x}^\mu \dot{x}_\mu} + \lambda\,\frac{d}{dt}\sqrt{\dot{x}^\mu \dot{x}_\mu} = \frac{d}{dt}\Big(\lambda\sqrt{\dot{x}^\mu \dot{x}_\mu}\Big) = \frac{d}{dt}\Big(\lambda L\Big)\end{aligned}$$

b) Noether identity: $\dfrac{\delta L}{\delta x^\mu}\,\dot{x}^\mu = 0.$

To verify this identity explicitly we first calculate $\frac{\delta L}{\delta x^\mu}$:

$$\frac{\delta L}{\delta x^\mu} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{d}{dt}\frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = -\frac{\ddot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} + \frac{\ddot{x}^\nu \dot{x}_\nu}{(\dot{x}^\varrho \dot{x}_\varrho)^{3/2}}\,\dot{x}_\mu\ .$$

This gives: $\dfrac{\delta L}{\delta x^\mu}\,\dot{x}^\mu = -\dfrac{\ddot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}\,\dot{x}^\mu + \dfrac{\ddot{x}^\nu \dot{x}_\nu}{(\dot{x}^\varrho \dot{x}_\varrho)^{3/2}}\,\dot{x}_\mu \dot{x}^\mu = -\dfrac{\ddot{x}_\mu \dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} + \dfrac{\ddot{x}^\nu \dot{x}_\nu}{(\dot{x}^\varrho \dot{x}_\varrho)^{1/2}} = 0.$

c) $J = \lambda L - (\delta_\lambda x^\mu)\dfrac{\partial L}{\partial \dot{x}^\mu} = \lambda\,\sqrt{\dot{x}^\mu \dot{x}_\mu} - \lambda\,\dot{x}^\mu\,\dfrac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = \lambda\,\sqrt{\dot{x}^\mu \dot{x}_\mu} - \lambda\,\dfrac{\dot{x}^\mu \dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = 0.$

Solution of exercise 15
δ_λ is a local symmetry because of $\delta_\lambda\, L = m\,\dfrac{d}{dt}\Big(\dot{\lambda}\dot{x}\ddot{x} + 2\lambda\dot{x}\,\dddot{x} - \lambda\ddot{x}^2\Big)$.
It is trivial because of $\delta_\lambda\, x = -2\lambda\dfrac{d}{dt}\,\dfrac{\delta L}{\delta x} - \dot{\lambda}\,\dfrac{\delta L}{\delta x} = -\dfrac{d}{dt}\Big(\lambda\,\dfrac{\delta L}{\delta x}\Big) + (-\lambda)\,\dfrac{d}{dt}\,\dfrac{\delta L}{\delta x}\ .$