3 Poincaré symmetry

3.1 Lorentz transformations

3.1.1 Lorentz transformations of the spacetime coordinates

Lorentz transformations of spacetime coordinates are denoted by

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu\, x^\nu .$$

Examples in four dimensions with $(x^0, x^1, x^2, x^3) = (t, x, y, z)$:
Boost in x-direction:

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{3.1}$$

Rotation around z-axis:

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{3.2}$$

The defining property of Lorentz transformations is that they leave invariant the Minkowski metric $\eta^{\mu\nu}$:

$$\eta^{\mu\nu} = \Lambda^\mu{}_\varrho \Lambda^\nu{}_\sigma \eta^{\varrho\sigma}. \tag{3.3}$$

We use the ‘mostly plus’ convention for the Minkowski metric (in all dimensions):

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{3.4}$$

Infinitesimal transformations ($\Lambda = \exp(-\varepsilon)$):

$$\begin{aligned} & & \Lambda^\mu{}_\nu &= \delta^\mu_\nu - \varepsilon^\mu{}_\nu + O(\varepsilon^2) \\ (3.3) & \quad\Rightarrow & \eta^{\mu\nu} &= (\delta^\mu_\varrho - \varepsilon^\mu{}_\varrho)(\delta^\nu_\sigma - \varepsilon^\nu{}_\sigma)\eta^{\varrho\sigma} + O(\varepsilon^2) \\ & & &= \eta^{\mu\nu} - \varepsilon^\mu{}_\varrho \eta^{\varrho\nu} - \varepsilon^\nu{}_\sigma \eta^{\mu\sigma} + O(\varepsilon^2) \end{aligned} \tag{3.5}$$

(3.5) imposes

$$\varepsilon^{\mu\nu} + \varepsilon^{\nu\mu} = 0 \quad \text{where} \quad \varepsilon^{\mu\nu} := \varepsilon^\mu{}_\varrho \eta^{\varrho\nu}$$

Hence, the parameters $\varepsilon^\mu{}_\nu$ of Lorentz transformations are antisymmetric when written with two upper (or lower) indices,

$$\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu} \quad \Rightarrow \quad \varepsilon^0{}_i = \varepsilon^i{}_0\,, \quad \varepsilon^i{}_j = -\varepsilon^j{}_i\,. \tag{3.6}$$

(3.1) is recovered by choosing $\varepsilon^\mu{}_\nu$ such that only $\varepsilon^0{}_1$ and $\varepsilon^1{}_0$ are different from zero:

$$\varepsilon^0{}_1 = \varepsilon^1{}_0 = a, \quad \varepsilon^\mu{}_\nu = 0 \quad \text{elsewise} \quad \Rightarrow \quad \Lambda = \exp(-\varepsilon) = \begin{pmatrix} \cosh a & -\sinh a & 0 & 0 \\ -\sinh a & \cosh a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Comparing this to (3.1) we find that $\tanh a = v/c$, i.e. $a = \operatorname{artanh}(v/c)$.

(3.2) is recovered by choosing $\varepsilon^\mu{}_\nu$ such that only $\varepsilon^1{}_2$ and $\varepsilon^2{}_1$ are different from zero:

$$\varepsilon^2{}_1 = -\varepsilon^1{}_2 = \alpha, \quad \varepsilon^\mu{}_\nu = 0 \quad \text{elsewise} \quad \Rightarrow \quad \Lambda = \exp(-\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the following we shall use units such that $c = 1$.

3.1.2 Lorentz transformations of fields

Scalar fields

Let us begin with the simplest case, a real scalar field $\varphi(x)$. It transforms according to

$$\tilde\varphi(\tilde x) = \varphi(x), \quad \tilde x^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{3.7}$$

To derive the infinitesimal transformations $\delta_\varepsilon\varphi$ we use $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \varepsilon^\mu{}_\nu + O(\varepsilon^2)$ which gives $\tilde x^\mu = x^\mu - \varepsilon^\mu{}_\nu x^\nu + O(\varepsilon^2)$. The Taylor expansion of $\tilde\varphi(\tilde x)$ gives to first order in ε

$$\tilde\varphi(\tilde x) = \tilde\varphi(x^\mu - \varepsilon^\mu{}_\nu x^\nu + O(\varepsilon^2)) = \tilde\varphi(x) - \varepsilon^\mu{}_\nu x^\nu \partial_\mu \varphi(x) + O(\varepsilon^2).$$

Inserting this into (3.7), we obtain for $\delta_\varepsilon\varphi(x)$ (i.e., for $\tilde\varphi(x) - \varphi(x)$ linear in ε):

$$\delta_\varepsilon\varphi = \varepsilon^\mu{}_\nu x^\nu \partial_\mu \varphi \ . \tag{3.8}$$

Contravariant vector fields

A contravariant vector field $A^\varrho(x)$ transforms according to

$$\tilde A^\varrho(\tilde x) = \Lambda^\varrho{}_\sigma A^\sigma(x), \quad \tilde x^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{3.9}$$

We Taylor expand $\tilde A^\rho(\tilde x)$ in ε as we expanded before a scalar field. Using $\Lambda^\varrho{}_\sigma = \delta^\sigma_\sigma - \varepsilon^\varrho{}_\sigma + O(\varepsilon^2)$, the expansion of $\Lambda^\varrho{}_\nu A^\nu(x)$ in (3.9) reads

$$\Lambda^\varrho{}_\sigma A^\sigma(x) = A^\varrho(x) - \varepsilon^\varrho{}_\sigma A^\sigma(x) + O(\varepsilon^2).$$

This gives the infinitesimal transformations

$$\delta_\varepsilon A^\varrho = \varepsilon^\mu{}_\nu x^\nu \partial_\mu A^\varrho - \varepsilon^\varrho{}_\sigma A^\sigma. \tag{3.10}$$

Covariant vector fields

A covariant vector field $A_\varrho(x)$ transforms according to

$$\tilde A_\varrho(\tilde x) = A_\sigma(x)(\Lambda^{-1})^\sigma{}_\varrho \ , \quad \tilde x^\mu = \Lambda^\mu{}_\nu x^\nu \tag{3.11}$$

where $(\Lambda^{-1})^\nu{}_\varrho = (\exp(\varepsilon))^\nu{}_\varrho = \delta^\nu_\varrho + \varepsilon^\nu{}_\varrho + O(\varepsilon^2)$ is the matrix inverse of $\Lambda^\mu{}_\varrho = (\exp(-\varepsilon))^\mu{}_\varrho$. Proceeding as in the case of the contravariant vector field, we obtain

$$\delta_\varepsilon A_\varrho = \varepsilon^\mu{}_\nu x^\nu \partial_\mu A_\varrho + A_\sigma \varepsilon^\sigma{}_\varrho \ . \tag{3.12}$$

Tensor fields

A tensor field $T^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}$ of type (r,s) carries r upper and s lower indices. Its finite transformations are

$$\tilde{T}^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}(\tilde{x}) = \Lambda^{\mu_1}{}_{\varrho_1}\cdots\Lambda^{\mu_r}{}_{\varrho_r}T^{\varrho_1\dots\varrho_r}_{\sigma_1\dots\sigma_s}(x)(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\cdots(\Lambda^{-1})^{\sigma_s}{}_{\nu_s}\ , \quad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \tag{3.13}$$

The corresponding infinitesimal transformations are

$$\delta_\varepsilon T^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} = \varepsilon^\varrho{}_\sigma x^\sigma\partial_\varrho T^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s} - \sum_{i=1}^{r}\varepsilon^{\mu_i}{}_\varrho T^{\mu_1\dots\mu_{i-1}\varrho\mu_{i+1}\dots\mu_r}_{\nu_1\dots\nu_s} + \sum_{i=1}^{s}T^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_{i-1}\varrho\nu_{i+1}\dots\nu_s}\varepsilon^\varrho{}_{\nu_i}\ . \tag{3.14}$$

General fields

In general, fields ϕ^i forming a linear representation of the Lorentz group transform according to

$$\tilde{\phi}^i(\tilde{x}) = \exp(-\tfrac{1}{2}\varepsilon^{\mu\nu}S_{\mu\nu})^i{}_j\phi^j(x), \quad \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{3.15}$$

where the $S_{\mu\nu}$ are contant matrices which represent the Lorentz algebra[5]:

$$S_{\mu\nu} = -S_{\nu\mu}, \quad [S_{\mu\nu}, S_{\varrho\sigma}] = \eta_{\nu\varrho}S_{\mu\sigma} - \eta_{\mu\varrho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\varrho} + \eta_{\mu\sigma}S_{\nu\varrho}\ . \tag{3.16}$$

The corresponding infinitesimal transformations are

$$\delta_\varepsilon\phi^i = \varepsilon^\mu{}_\nu x^\nu\partial_\mu\phi^i - \tfrac{1}{2}\,\varepsilon^{\mu\nu}S_{\mu\nu}{}^i{}_j\phi^j. \tag{3.17}$$

Comments: Scalar fields fulfill (3.15) and (3.17) with $S_{\mu\nu} = 0$. The representation matrices $S_{\mu\nu}$ of the Lorentz algebra arising from (3.10) are subject of exercise 11b. The representation matrices arising from (3.12) are obtained analogously but one has to be careful with regard to indices[6]. The representation matrices $S_{\mu\nu}$ arising from (3.14) can be obtained by ordering the components of $T^{\mu_1\dots\mu_r}_{\nu_1\dots\nu_s}$ in some way (e.g., for $n = 2$ and $(r,s) = (2,0)$ one may choose $(T^{00}, T^{01}, T^{10}, T^{11}) \equiv (\phi^1, \phi^2, \phi^3, \phi^4)$). Spinor fields will be discussed next.

Spinor fields

Spinor fields transform under the Lorentz group according to the spin representation of the Lorentz algebra (3.31). This representation can be constructed by means of a set of matrices Γ_μ, $\mu = 0,\dots,n-1$. For $n = 2k$ and $n = 2k+1$, these are $2^k\times 2^k$ matrices with complex entries which fulfill the Dirac algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1} \tag{3.18}$$

where $\{\ ,\ \}$ denotes the anticommutator ($\{M,N\} := MN + NM$), and $\mathbf{1}$ is the $2^k\times 2^k$ unit matrix. More explicitly, (3.18) reads

$$\Gamma_0\Gamma_0 = -\mathbf{1}, \quad \Gamma_i\Gamma_i = \mathbf{1}\ \text{(no sum over } i \text{ here)}, \quad \mu\neq\nu:\ \Gamma_\mu\Gamma_\nu = -\Gamma_\nu\Gamma_\mu\ . \tag{3.19}$$

In four dimensions, a set of matrices fulfilling (3.18) is

$$\Gamma_0 = \mathrm{i}\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}, \quad \Gamma_i = \mathrm{i}\begin{pmatrix}\mathbf{0} & -\sigma_i\\ \sigma_i & \mathbf{0}\end{pmatrix} \tag{3.20}$$

[5] We are using matrix notion. $S_{\mu\nu}{}^i{}_j$ are the entries of the matrix $S_{\mu\nu}$ (i and j label the rows and columns, respectively). $S_{\mu\nu}S_{\varrho\sigma}$ denotes the matrix product of $S_{\mu\nu}$ and $S_{\varrho\sigma}$, i.e. $(S_{\mu\nu}S_{\varrho\sigma})^i{}_j = S_{\mu\nu}{}^i{}_k S_{\varrho\sigma}{}^k{}_j$.

[6] To match (3.12) with (3.17), one has to make the index identifications $\varrho\equiv i$ and $\sigma\equiv j$.

where the **0** and **1** denote the 2×2 zero and unit matrix respectively, and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.21}$$

A direct consequence of (3.18) is that the matrices

$$\Sigma_{\mu\nu} := \tfrac{1}{4}\,[\Gamma_\mu, \Gamma_\nu] \tag{3.22}$$

form a representation of the Lorentz algebra. To verify this we proceed as follows:

$$\begin{aligned}
[\Gamma_\mu\Gamma_\nu, \Gamma_\varrho] &= \Gamma_\mu\Gamma_\nu\Gamma_\varrho - \Gamma_\varrho\Gamma_\mu\Gamma_\nu \\
&= \Gamma_\mu(\{\Gamma_\nu, \Gamma_\varrho\} - \Gamma_\varrho\Gamma_\nu) - (\{\Gamma_\varrho, \Gamma_\mu\} - \Gamma_\mu\Gamma_\varrho)\Gamma_\nu \\
&= 2\eta_{\nu\varrho}\Gamma_\mu - 2\eta_{\varrho\mu}\Gamma_\nu
\end{aligned} \tag{3.23}$$

$$\Rightarrow \quad [\Sigma_{\mu\nu}, \Gamma_\varrho] = \tfrac{1}{4}\,[\Gamma_\mu\Gamma_\nu, \Gamma_\varrho] - (\mu \leftrightarrow \nu) \stackrel{(3.23)}{=} \eta_{\nu\varrho}\Gamma_\mu - \eta_{\mu\varrho}\Gamma_\nu \tag{3.24}$$

$$\begin{aligned}
\Rightarrow \quad [\Sigma_{\mu\nu}, \Gamma_\varrho\Gamma_\sigma] &= [\Sigma_{\mu\nu}, \Gamma_\varrho]\Gamma_\sigma + \Gamma_\varrho[\Sigma_{\mu\nu}, \Gamma_\sigma] \\
&\stackrel{(3.24)}{=} \eta_{\nu\varrho}\Gamma_\mu\Gamma_\sigma - \eta_{\mu\varrho}\Gamma_\nu\Gamma_\sigma + \eta_{\nu\sigma}\Gamma_\varrho\Gamma_\mu - \eta_{\mu\sigma}\Gamma_\varrho\Gamma_\nu
\end{aligned} \tag{3.25}$$

$$\Rightarrow \quad [\Sigma_{\mu\nu}, \Sigma_{\varrho\sigma}] \stackrel{(3.25)}{=} \eta_{\nu\varrho}\Sigma_{\mu\sigma} - \eta_{\mu\varrho}\Sigma_{\nu\sigma} + \eta_{\nu\sigma}\Sigma_{\varrho\mu} - \eta_{\mu\sigma}\Sigma_{\varrho\nu}\ . \tag{3.26}$$

In $n = 2k$ and $n = 2k+1$ dimensions the $\Sigma_{\mu\nu}$ are $2^k \times 2^k$ matrices and therefore act on spinor fields $\psi(x)$ with 2^k components (ψ^α, $\alpha = 1, \ldots, 2^k$). Lorentz transformations of spinor fields are generated according to (3.15) and (3.17), with representation matrices $S_{\mu\nu}$ given by $\Sigma_{\mu\nu}$. Using matrix notation, with ψ a 'column spinor' on which $\Sigma_{\mu\nu}$ acts, we obtain:

$$\tilde{\psi}(\tilde{x}) = \exp(-\tfrac{1}{2}\varepsilon^{\mu\nu}\Sigma_{\mu\nu})\psi(x), \tag{3.27}$$

$$\delta_\varepsilon\psi = \varepsilon^\mu{}_\nu x^\nu \partial_\mu\psi - \tfrac{1}{2}\,\varepsilon^{\mu\nu}\Sigma_{\mu\nu}\psi, \quad \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \end{pmatrix}. \tag{3.28}$$

Remarks: The set of matrices $\{\Gamma_\mu\}$ is by no means unique. Indeed, when $\{\Gamma_\mu\}$ is a set of $2^k \times 2^k$ matrices fulfilling (3.18), then $\{M\Gamma_\mu M^{-1}\}$ also fulfills (3.18) for every invertible $2^k \times 2^k$ matrix M. Hence, there is a huge freedom to choose Γ-matrices. One can show that, in even dimension $n = 2k$, the $2^k \times 2^k$ Γ-matrices are unique up to equivalence transformations $M\Gamma_\mu M^{-1}$. Furthermore, one can always choose a set of unitary Γ-matrices. With no loss of generality we will therefore assume in the following that all the Γ-matrices are unitary, i.e.,

$$\forall\, \mu: \quad {\Gamma_\mu}^\dagger\,\Gamma_\mu = \mathbf{1}, \tag{3.29}$$

where ${\Gamma_\mu}^\dagger = {\Gamma_\mu}^{*\top}$ denotes the complex conjugated and transposed matrix Γ_μ.

3.2 Lorentz algebra

From (3.17) one reads off the infinitesimal Lorentz transformations $M_{\mu\nu}$ corresponding to $\varepsilon^{\mu\nu}$:

$$\delta_\varepsilon\phi^i = \tfrac{1}{2}\,\varepsilon^{\mu\nu}M_{\mu\nu}\phi^i, \quad M_{\mu\nu}\phi^i = \underbrace{(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi^i}_{\text{orbital part}}\underbrace{-S_{\mu\nu}{}^i{}_j\phi^j}_{\text{spin part}}\ . \tag{3.30}$$

They generate boosts and spatial rotations (cf. section 3.1.1),

$$M_{0i}: \ \text{boost in } x^i \text{ direction}, \quad M_{ij}: \ \text{rotation in the } x^i\text{-}x^j \text{ plane},$$

and fulfill the Lorentz algrebra (cf. exercise 11a):

$$[M_{\mu\nu}, M_{\varrho\sigma}] = \eta_{\nu\varrho} M_{\mu\sigma} - \eta_{\mu\varrho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\varrho} + \eta_{\mu\sigma} M_{\nu\varrho} \; . \tag{3.31}$$

This gives

$$[M_{0i}, M_{0j}] = M_{ij} \; , \tag{3.32}$$
$$[M_{0i}, M_{jk}] = \delta_{ij} M_{0k} - \delta_{ik} M_{0j} \; , \tag{3.33}$$
$$[M_{ij}, M_{kl}] = \delta_{il} M_{jk} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik} + \delta_{jk} M_{il} \; . \tag{3.34}$$

According to (3.32), the commutator of two infinitesimal boosts in two different directions is an infinitesimal rotation in the plane spanned by these directions. Equation (3.33) shows that the commutator of an infinitesimal boost in x^j direction and an infinitesimal rotation in the x^j-x^k plane is an infinitesimal boost in x^k direction. Equation (3.34) is the commutator algebra of infinitesimal spatial rotations. In four dimensional spacetime it is the familiar angular momentum algebra of three dimensional space (using $M_{12} = -L_3$ etc., (3.34) becomes $[L_1, L_2] = L_3$ etc.).

3.3 Poincaré transformations and algebra

Poincaré transformations contain spacetime translations in addition to Lorentz transformations. Spacetime translations of the coordinates are given by $\tilde{x}^\mu = x^\mu - \varepsilon^\mu$ with constant ε^μ. The corresponding transformations of fields are $\tilde{\phi}^i(x^\mu - \varepsilon^\mu) = \phi^i(x)$. Hence infinitesimal Poincaré transformations of fields read

$$\delta_\varepsilon \phi^i = (\varepsilon^\mu + \varepsilon^\mu{}_\nu x^\nu)\partial_\mu \phi^i - \tfrac{1}{2}\, \varepsilon^{\mu\nu} S_{\mu\nu}{}^i{}_j \phi^j. \tag{3.35}$$

Analogously to (3.30) we read off from (3.35) the infinitesimal Poincaré transformations corresponding to ε^μ and $\varepsilon^{\mu\nu}$, respectively:

$$\delta_\varepsilon \phi^i = (\varepsilon^\mu P_\mu + \tfrac{1}{2}\, \varepsilon^{\mu\nu} M_{\mu\nu})\phi^i, \quad P_\mu \phi^i = \partial_\mu \phi^i, \quad M_{\mu\nu}\phi^i = (x_\nu \partial_\mu - x_\mu \partial_\nu)\phi^i - S_{\mu\nu}{}^i{}_j \phi^j. \tag{3.36}$$

The commutators of these transformations fulfill the Poincaré algebra (cf. exercise 11a)

$$[P_\mu, P_\nu] = 0, \tag{3.37}$$
$$[M_{\mu\nu}, P_\varrho] = \eta_{\varrho\nu} P_\mu - \eta_{\varrho\mu} P_\nu \, , \tag{3.38}$$
$$[M_{\mu\nu}, M_{\varrho\sigma}] = \eta_{\nu\varrho} M_{\mu\sigma} - \eta_{\mu\varrho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\varrho} + \eta_{\mu\sigma} M_{\nu\varrho} \; . \tag{3.39}$$

3.4 Poincaré invariant actions

The method to construct Poincaré invariant actions is to construct Lagrangians L that transform scalarly under infinitesimal Poincaré transformations, i.e.

$$\delta_\varepsilon L = (\varepsilon^\mu + \varepsilon^\mu{}_\nu x^\nu)\partial_\mu L = \partial_\mu(\varepsilon^\mu L + \varepsilon^\mu{}_\nu x^\nu L). \tag{3.40}$$

The second equality holds owing to $\partial_\mu(\varepsilon^\mu{}_\nu x^\nu) = \varepsilon^\mu{}_\mu = 0$ (one has $\varepsilon^\mu{}_\mu = \varepsilon^{\mu\nu}\eta_{\nu\mu} = 0$ because of $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$), and ensures that Poincaré transformations generate a symmetry of the action $S = \int d^n x\, L$.

Scalar Lagrangians are constructed by 'properly contracting' all Lorentz vector and spinor indices so that the resultant Lagrangian does not carry free indices of these types. Indices are 'properly contracted' in a convenient way by means of (numerically) invariant tensors. These are constant tensors that are left invariant by Poincaré transformations, i.e., they have vanishing δ_ε-transformations (3.35). We know already an invariant tensor, namely the Minkowski metric $\eta^{\mu\nu}$ (recall that, by definition, Lorentz transformations leave $\eta^{\mu\nu}$ invariant, cf. (3.3)). Another invariant tensor is the Levi-Civita tensor $\epsilon^{\mu_1\cdots\mu_n}$ (also called permutation tensor) which is totally antisymmetric and defined according to

$$\epsilon^{0\ldots n-1} = 1, \quad \epsilon^{\mu_1\cdots\mu_i\cdots\mu_j\cdots\mu_n} = -\epsilon^{\mu_1\cdots\mu_j\cdots\mu_i\cdots\mu_n} \quad \forall i,j\ (i \neq j). \tag{3.41}$$

Actually, $\epsilon^{\mu_1\cdots\mu_n}$ is invariant only under Lorentz transformations that are continuously connected to the identity (so-called proper and orthochronous Lorentz transformations) and these are the only ones we consider here[7]. Let us verify explicitly that $\epsilon^{\mu_1\cdots\mu_n}$ is an invariant tensor under proper and orthochronous Lorentz transformations (since the components of $\epsilon^{\mu_1\cdots\mu_n}$ are constant they are clearly invariant under spacetime translations). To this end we apply (3.14) to $\epsilon^{\mu_1\cdots\mu_n}$. We obtain

$$\delta_\varepsilon \epsilon^{\mu_1\cdots\mu_n} = -\sum_{i=1}^{n} \varepsilon^{\mu_i}{}_{\varrho}\, \epsilon^{\mu_1\cdots\mu_{i-1}\varrho\mu_{i+1}\cdots\mu_n}.$$

Owing to (3.41), the r.h.s. of the latter equation is totally antisymmetric in $\mu_1\ldots\mu_n$. Hence, it is proportional to $\epsilon^{\mu_1\cdots\mu_n}$ (in n dimensions a totally antisymmetric object $T^{\mu_1\cdots\mu_n}$ fulfills $T^{\mu_1\cdots\mu_n} = T^{0\ldots n-1}\epsilon^{\mu_1\cdots\mu_n}$). Contracting the expression on the r.h.s. with $\epsilon_{\mu_1\ldots\mu_n}$, one obtains that the proportionality constant is $-\varepsilon^{\mu}{}_{\mu}$ which vanishes owing to $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$. Hence,

$$\delta_\varepsilon \epsilon^{\mu_1\cdots\mu_n} = 0. \tag{3.42}$$

Further important invariant tensors under proper and orthochronous Lorentz transformations are the Γ-matrices. Recall that Γ_μ is a $2^k \times 2^k$ matrix (for $n = 2k$ or $n = 2k+1$). We denote its entries by $\Gamma_\mu{}^\alpha{}_\beta$. The positions of the spinor indices α and β indicate how Lorentz transformations act on these indices: α is treated as the index of the spinor ψ in (3.28), β as the index of a spinor $\bar\psi$ transforming contragrediently to ψ according to $\delta_\varepsilon\bar\psi = \varepsilon^\mu{}_\nu x^\nu\partial_\mu\bar\psi + \frac{1}{2}\varepsilon^{\varrho\sigma}\bar\psi\Sigma_{\varrho\sigma}$ (see section 3.4.3 for how such spinors are constructed)[8]. The index μ of Γ_μ is transformed as a covariant vector index (we use greek letters from the beginning of the alphabet to denote spinor indices whereas greek letters from the middle or end of the alphabet denote vector indices). This gives

$$\delta_\varepsilon\Gamma_\mu{}^\alpha{}_\beta = \varepsilon^\varrho{}_\mu\Gamma_\varrho{}^\alpha{}_\beta - \tfrac{1}{2}\varepsilon^{\varrho\sigma}\Sigma_{\varrho\sigma}{}^\alpha{}_\gamma\Gamma_\mu{}^\gamma{}_\beta + \tfrac{1}{2}\varepsilon^{\varrho\sigma}\Gamma_\mu{}^\alpha{}_\gamma\Sigma_{\varrho\sigma}{}^\gamma{}_\beta.$$

Returning to matrix notation and suppressing the writing of spinor indices, the r.h.s. of the latter equation reads $\varepsilon^\varrho{}_\mu\Gamma_\varrho - \frac{1}{2}\varepsilon^{\varrho\sigma}[\Sigma_{\varrho\sigma},\Gamma_\mu]$. Using (3.24) one easily verifies that this vanishes. Hence,

$$\delta_\varepsilon\Gamma_\mu{}^\alpha{}_\beta = 0. \tag{3.43}$$

Equations (3.42) and (3.43) are to be interpreted with care because $\epsilon^{\mu_1\cdots\mu_n}$ and $\Gamma_\mu{}^\alpha{}_\beta$ are constants (pure numbers) and constants are not transformed when inspecting symmetries of the type we are interested in (e.g., in $L = \frac{m}{2}\dot q^2$ we do not transform $\frac{m}{2}$). What equations (3.42) and (3.43) actually mean is that the constancy of these objects is compatible with Poincaré transformations. That is, when discussing Poincaré symmetry we are allowed to treat these particular objects (and, more generally, all Poincaré invariant tensors) as fields which transform according to their indices, in spite of the fact that they are not fields because it makes

[7]In addition there are Lorentz transformations that contain the space inversion $P: t \mapsto t, x^i \mapsto -x^i$ or the time reversal $T: t \mapsto -t, x^i \mapsto x^i$ (or both P and T). $\epsilon^{\mu_1\cdots\mu_n}$ is not invariant under T, under P only if n is odd, and under PT only if n is even. Any Lorentz transformation is either proper and orthochronous, or it is the product of a proper and orthochronous Lorentz transformation with P, T or PT.

[8]Our conventions are such that Γ_μ acts on ψ from the left according to $(\Gamma_\mu\psi)^\alpha = \Gamma_\mu{}^\alpha{}_\beta\psi^\beta$ which shows that the α-index of $\Gamma_\mu{}^\alpha{}_\beta$ is to be treated like the indices of ψ while the β-indices are contracted with the indices of ψ and should therefore be treated as the indices of $\bar\psi$.

no difference whether or not they are transformed. Viewing them as transforming objects is particularly convenient for the construction of invariant actions because one can immediately read off the transformation properties of functions composed of fields and invariant tensors from the index structure.

3.4.1 Free scalar field

The infinitesimal Poincaré transformations of a scalar field have been given in (3.8). The transformation of the spacetime derivatives of a scalar field are obtained by applying ∂_μ to (3.8):

$$\delta_\varepsilon \partial_\mu \varphi = \partial_\mu(\delta_\varepsilon \varphi) = \partial_\mu(\varepsilon^\varrho{}_\nu x^\nu \partial_\varrho \varphi) = \varepsilon^\varrho{}_\nu x^\nu \partial_\varrho \partial_\mu \varphi + \varepsilon^\varrho{}_\mu \partial_\varrho \varphi \ . \tag{3.44}$$

This shows that $\partial_\mu \varphi$ is a covariant vector field (cf. (3.12)). As a consequence, the product $\partial_\mu \varphi \partial_\nu \varphi$ is a (0,2)-tensor field, i.e., it transforms according to (3.14) with $(r, s) = (0, 2)$. Hence, contraction of $\partial_\mu \varphi \partial_\nu \varphi$ with $\eta^{\mu\nu}$ results in a scalar quantity and the following Lagrangian transforms scalarly under Poincaré transformations (the reader is invited to check this explicitly):

$$L([\varphi]) = -\frac{1}{2}\, \varphi_{,\mu} \varphi_{,\nu} \eta^{\mu\nu} - \frac{m^2}{2}\, \varphi^2. \tag{3.45}$$

The corresponding Poincaré invariant action is $S[\varphi] = \int d^n x\, L([\varphi])$.

3.4.2 Free vector (gauge) field

The infinitesimal Poincaré transformations of a covariant vector field have been given in (3.12). Analogously to (3.44) one verifies that the following so-called field strength tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ of A_μ are tensor fields of type (0,2) and (2,0), respectively:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \quad F^{\mu\nu} = \eta^{\mu\varrho} \eta^{\nu\sigma} F_{\varrho\sigma} \ . \tag{3.46}$$

An appropriate Lagrangian for A_μ which transforms scalarly under Poincaré transformations is

$$L([A]) = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu}. \tag{3.47}$$

The corresponding Poincaré invariant action is $S[\varphi] = \int d^n x\, L([A])$. This action is gauge invariant, as we shall discuss later.

3.4.3 Free spinor field

To construct Poincaré invariant actions for a spinor field ψ transforming according to (3.28) one uses that the spinor field

$$\bar\psi := \psi^\dagger \Gamma_0, \quad \psi^\dagger = (\psi^{1*}, \psi^{2*}, \dots) \tag{3.48}$$

transforms 'contragrediently' to ψ under Poincaré transformations, i.e., according to

$$\delta_\varepsilon \bar\psi = \varepsilon^\varrho{}_\sigma x^\sigma \partial_\varrho \bar\psi + \tfrac{1}{2} \varepsilon^{\varrho\sigma} \bar\psi \Sigma_{\varrho\sigma} \ . \tag{3.49}$$

The notation $\psi^\dagger = (\psi^{1*}, \psi^{2*}, \dots)$ in (3.48) indicates that $\psi^\dagger$ in (3.48) is viewed as a 'row spinor' (in contrast to the 'column spinor' ψ in (3.28)) whose components are the complex conjugated components of ψ. Accordingly, $\bar\psi = (\bar\psi^1, \bar\psi^2, \dots)$ in (3.49) is viewed as a row spinor too. The derivation of (3.49) is the subject of exercise 12. Here we only note that it follows from

$$\forall\,\mu: \quad {\Gamma_\mu}^\dagger = \Gamma_0\Gamma_\mu\Gamma_0\ , \tag{3.50}$$

which itself is a consequence of the Dirac algebra (3.18) with Minkowski metric as in (3.4) and the unitarity of the Γ-matrices (3.29) that we have assumed with no loss of generality[9].

By means of $\bar\psi$ one constructs the following Lagrangian:

$$L([\psi]) = \mathrm{i}\,\bar\psi\Gamma^\mu\psi_{,\mu} + \mathrm{i}\,m\bar\psi\psi, \tag{3.51}$$

where i is the imaginary unit and m is a real mass parameter. Using (3.28), (3.49) and (3.24) it is straightforward to verify that $L([\psi])$ transforms scalarly under Poincaré transformations (alternatively, one can conclude this using that the Γ-matrices are Poincaré invariant tensors). Hence, $S[\psi] = \int d^n x\, L([\psi])$ is a Poincaré invariant action.

The factors i in (3.51) ensure that the Lagrangian is real up to a total divergence. To show this we compute the complex conjugates of $\bar\psi\psi$ and $\bar\psi\Gamma^\mu\psi_{,\mu}$:

$$(\bar\psi\psi)^* = (\bar\psi\psi)^\dagger = (\psi^\dagger\Gamma_0\psi)^\dagger = \psi^\dagger{\Gamma_0}^\dagger\psi^{\dagger\dagger} = \psi^\dagger(-\Gamma_0)\psi = -\bar\psi\psi, \tag{3.52}$$

$$\begin{aligned}(\bar\psi\Gamma^\mu\psi_{,\mu})^* &= (\psi^\dagger\Gamma_0\Gamma^\mu\psi_{,\mu})^\dagger = \psi^\dagger_{,\mu}\Gamma^{\mu\dagger}{\Gamma_0}^\dagger\psi^{\dagger\dagger} = \psi^\dagger_{,\mu}(\Gamma_0\Gamma^\mu\Gamma_0)(-\Gamma_0)\psi\\ &= (\partial_\mu\psi^\dagger)\Gamma_0\Gamma^\mu\psi = \partial_\mu(\psi^\dagger\Gamma_0\Gamma^\mu\psi) - \psi^\dagger\Gamma_0\Gamma^\mu\partial_\mu\psi = \partial_\mu(\bar\psi\Gamma^\mu\psi) - \bar\psi\Gamma^\mu\psi_{,\mu}\ .\end{aligned} \tag{3.53}$$

(3.52) shows that $\bar\psi\psi$ is purely imaginary. Hence, $\mathrm{i}m\bar\psi\psi$ is real. According to (3.53), $\bar\psi\Gamma^\mu\psi_{,\mu}$ is purely imaginary up to the (real) total divergence $\partial_\mu(\bar\psi\Gamma^\mu\psi)$. Hence, $\mathrm{i}\bar\psi\Gamma^\mu\psi_{,\mu}$ is real up to the total divergence $\partial_\mu(\mathrm{i}\bar\psi\Gamma^\mu\psi)$. A truly real Lagrangian is obtained by using $\frac{\mathrm{i}}{2}\bar\psi\Gamma^\mu\psi_{,\mu} - \frac{\mathrm{i}}{2}\bar\psi_{,\mu}\Gamma^\mu\psi$ in place of $\mathrm{i}\bar\psi\Gamma^\mu\psi_{,\mu}$.

Remark: Whether or not there is factor i in front of the 'mass term' $m\bar\psi\psi$ in (3.51) depends on the conventions. Indeed, choosing a Minkowski metric with signature $(+-\cdots-)$ in place of $(-+\cdots+)$, equations (3.48)–(3.50) still hold for unitary Γ-matrices, but in place of (3.52) one obtains $(\bar\psi\psi)^* = \bar\psi\psi$ because one has ${\Gamma_0}^\dagger = {\Gamma_0}^{-1} = \Gamma_0$ rather than ${\Gamma_0}^\dagger = {\Gamma_0}^{-1} = -\Gamma_0$ (because the Dirac algebra gives in this case $\Gamma_0\Gamma_0 = \mathbf{1}$).

3.5 Conserved currents and tensors

Using equation (2.6) it is straightforward to determine the conserved currents which correspond to Poincaré symmetries when the Lagrangian transforms scalarly as in (3.40) and depends only on the fields and their first order derivatives (for Lagrangians depending also on higher order derivatives equation (2.6) gets replaced by a more complicated expression). Equation (3.40) gives $\delta_\varepsilon L = \partial_\mu K^\mu$ with $K^\mu = \varepsilon^\mu L + \varepsilon^\mu{}_\nu x^\nu L$. Using this in equation (2.6), we obtain the conserved currents

$$J^\mu = \varepsilon^\mu L + \varepsilon^\mu{}_\sigma x^\sigma L - (\delta_\varepsilon\phi^i)\,\frac{\partial L}{\partial\phi^i_{,\mu}}\ . \tag{3.54}$$

Using now equation (3.35) in equation (3.54), the latter becomes

$$J^\mu = \varepsilon^\mu L + \varepsilon^\mu{}_\sigma x^\sigma L - (\varepsilon^\nu\phi^i_{,\nu} + \varepsilon^\varrho{}_\sigma x^\sigma\phi^i_{,\varrho} - \tfrac{1}{2}\,\varepsilon^{\varrho\sigma}S_{\varrho\sigma}{}^i{}_j\phi^j)\,\frac{\partial L}{\partial\phi^i_{,\mu}}\ . \tag{3.55}$$

These currents are conserved for every choice of ε^μ and $\varepsilon^{\varrho\sigma}$. To extract the conserved quantities corresponding to ε^μ and $\varepsilon^{\varrho\sigma}$ respectively, we pick in (3.55) the terms involving ε^μ and $\varepsilon^{\varrho\sigma}$:

$$J^\mu = -\varepsilon^\nu T_\nu{}^\mu + \tfrac{1}{2}\,\varepsilon^{\varrho\sigma}J_{\varrho\sigma}{}^\mu, \tag{3.56}$$

[9] In the general case, without referring to a particular signature of the Minkowski metric or assuming unitarity of the Γ-matrices, one has $\bar\psi = \psi^\dagger A$ where A relates the matrices Γ_μ to ${\Gamma_\mu}^\dagger$ or $-{\Gamma_\mu}^\dagger$ according to ${\Gamma_\mu}^\dagger = A\Gamma_\mu A^{-1}$ or $-{\Gamma_\mu}^\dagger = A\Gamma_\mu A^{-1}$. (3.50) represents the case $-{\Gamma_\mu}^\dagger = A\Gamma_\mu A^{-1}$ with $A = \Gamma_0$.

$$T_\nu{}^\mu = \phi^i_{,\nu}\frac{\partial L}{\partial \phi^i_{,\mu}} - \delta^\mu_\nu L\,, \tag{3.57}$$

$$\begin{aligned} J_{\varrho\sigma}{}^\mu &= (\delta^\mu_\varrho x_\sigma - \delta^\mu_\sigma x_\varrho)L - (x_\sigma \phi^i_{,\varrho} - x_\varrho \phi^i_{,\sigma} - S_{\varrho\sigma}{}^i{}_j\phi^j)\frac{\partial L}{\partial \phi^i_{,\mu}} \\ &= x_\varrho T_\sigma{}^\mu - x_\sigma T_\varrho{}^\mu + S_{\varrho\sigma}{}^i{}_j\phi^j\frac{\partial L}{\partial \phi^i_{,\mu}}\ . \end{aligned} \tag{3.58}$$

Since the current J^μ is conserved for every choice of ε^μ and $\varepsilon^{\varrho\sigma}$, $T_\nu{}^\mu$ and $J_{\varrho\sigma}{}^\mu$ provide a conserved current for every choice of indices ν and $[\varrho\sigma]$:

$$\forall\,\nu:\ \partial_\mu T_\nu{}^\mu \approx 0, \quad \forall\,[\varrho\sigma]:\ \partial_\mu J_{\varrho\sigma}{}^\mu \approx 0. \tag{3.59}$$

$T_\nu{}^\mu$ is called the energy-momentum tensor.

3.6 Exercises 11–13

Exercise 11: Poincaré algebra
a) The infinitesimal Poincaré transformations of a scalar field φ read

$$\delta_\varepsilon\varphi = (\varepsilon^\mu P_\mu + \tfrac{1}{2}\varepsilon^{\mu\nu}M_{\mu\nu})\varphi, \quad P_\mu\varphi = \partial_\mu\varphi, \quad M_{\mu\nu}\varphi = (x_\nu\partial_\mu - x_\mu\partial_\nu)\varphi.$$

Verify that

$$\begin{aligned} [M_{\mu\nu}, P_\varrho]\varphi &= (\eta_{\varrho\nu}P_\mu - \eta_{\varrho\mu}P_\nu)\varphi, \\ [M_{\mu\nu}, M_{\varrho\sigma}]\varphi &= (\eta_{\nu\varrho}M_{\mu\sigma} - \eta_{\mu\varrho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\varrho} + \eta_{\mu\sigma}M_{\nu\varrho})\varphi. \end{aligned}$$

b) The infinitesimal Lorentz transformations of a contravariant vector field V^μ read

$$\delta_\varepsilon V^\varrho = \varepsilon^\mu{}_\nu x^\nu\partial_\mu V^\varrho - \varepsilon^\varrho{}_\sigma V^\sigma.$$

Determine the corresponding spin matrices $S_{\mu\nu}$ with entries $S_{\mu\nu}{}^\varrho{}_\sigma$ by rewriting $-\varepsilon^\varrho{}_\sigma V^\sigma$ in the form $-\frac{1}{2}\varepsilon^{\mu\nu}S_{\mu\nu}{}^\varrho{}_\sigma V^\sigma$ (mind that $S_{\mu\nu} \stackrel{!}{=} -S_{\nu\mu}$). Verify that the matrices $S_{\mu\nu}$ fulfill equation (3.16) and write explicitly the matrices S_{01} and S_{12} in four dimensions.

Exercise 12: Dirac adjoint spinor field
Assume that $\{\Gamma_0,\dots,\Gamma_{n-1}\}$ is a unitary representation of the Dirac algebra, i.e., $\Gamma_\mu{}^\dagger = \Gamma_\mu{}^{-1}$ for all μ, and $\{\Gamma_\mu,\Gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1}$, with Minkowski metric as in (3.4). Furthermore let $M^s_{\mu\nu}$ denote the spin part of $M_{\mu\nu}$ ($M^s_{\mu\nu}\phi^i = -S_{\mu\nu}{}^i{}_j\phi^j$). Show that this implies (with $\bar\psi = \psi^\dagger\Gamma_0$, $\Sigma_{\mu\nu} = \frac{1}{4}[\Gamma_\mu,\Gamma_\nu]$, $M^s_{\mu\nu}\psi = -\Sigma_{\mu\nu}\psi$):

a) $\forall\,\mu:\ \Gamma_0\Gamma_\mu\Gamma_0 = \Gamma_\mu{}^\dagger$

b) $\Sigma_{\mu\nu}{}^\dagger = \Gamma_0\Sigma_{\mu\nu}\Gamma_0$ (hint: use the result of a)

c) $M^s_{\mu\nu}\psi^\dagger = -\psi^\dagger\Gamma_0\Sigma_{\mu\nu}\Gamma_0$ (hint: use $M^s_{\mu\nu}\psi^* = (M^s_{\mu\nu}\psi)^*$ and the result of b)

d) $M^s_{\mu\nu}\bar\psi = \bar\psi\Sigma_{\mu\nu}$ (hint: use the result of c)

Exercise 13: Energy-momentum tensor for a scalar field
Determine the energy-momentum tensor (3.57) for the Lagrangian (3.45) and show $T_0{}^0 \geq 0$.

Solution of exercise 11
a) The general rules $[\delta_\varepsilon,\partial_\mu] = 0$ and $\delta_\varepsilon x^\mu = 0$ imply $[M_{\mu\nu},\partial_\varrho] = 0$, $[P_\mu,\partial_\nu] = 0$, $M_{\mu\nu}x^\varrho = 0$ and $P_\mu x^\nu = 0$. Furthermore one has $\partial_\mu x_\nu = \partial_\mu(\eta_{\nu\varrho}x^\varrho) = \eta_{\nu\varrho}\delta^\varrho_\mu = \eta_{\nu\mu}$. This gives:

$$[M_{\mu\nu}, P_\varrho]\varphi = M_{\mu\nu}\partial_\varrho\varphi - P_\varrho(x_\nu\partial_\mu - x_\mu\partial_\nu)\varphi$$

$$
\begin{aligned}
&= \partial_\varrho M_{\mu\nu}\varphi - (x_\nu\partial_\mu - x_\mu\partial_\nu)P_\varrho\varphi \\
&= \partial_\varrho(x_\nu\partial_\mu - x_\mu\partial_\nu)\varphi - (x_\nu\partial_\mu - x_\mu\partial_\nu)\partial_\varrho\varphi \\
&= (\eta_{\nu\varrho}\partial_\mu - \eta_{\mu\varrho}\partial_\nu)\varphi = (\eta_{\nu\varrho}P_\mu - \eta_{\mu\varrho}P_\nu)\varphi \\
[M_{\mu\nu}, M_{\varrho\sigma}]\varphi &= M_{\mu\nu}(x_\sigma\partial_\varrho - x_\varrho\partial_\sigma)\varphi - (\mu\leftrightarrow\varrho, \nu\leftrightarrow\sigma) \\
&= (x_\sigma\partial_\varrho - x_\varrho\partial_\sigma)M_{\mu\nu}\varphi - (\mu\leftrightarrow\varrho, \nu\leftrightarrow\sigma) \\
&= (x_\sigma\partial_\varrho - x_\varrho\partial_\sigma)(x_\nu\partial_\mu - x_\mu\partial_\nu)\varphi - (\mu\leftrightarrow\varrho, \nu\leftrightarrow\sigma) \\
&= \eta_{\nu\varrho}(x_\sigma\partial_\mu - x_\nu\partial_\sigma)\varphi + \dots = \eta_{\nu\varrho}M_{\mu\sigma}\varphi + \dots
\end{aligned}
$$

b) One has

$$
\begin{aligned}
&-\varepsilon^\varrho{}_\sigma V^\sigma = -\varepsilon^{\varrho\nu}\eta_{\nu\sigma}V^\sigma = -\varepsilon^{\mu\nu}\delta^\varrho_\mu\eta_{\nu\sigma}V^\sigma = -\tfrac{1}{2}\varepsilon^{\mu\nu}(\delta^\varrho_\mu\eta_{\nu\sigma} - \delta^\varrho_\nu\eta_{\mu\sigma})V^\sigma \\
&\Rightarrow\quad S_{\mu\nu}{}^\varrho{}_\sigma = \delta^\varrho_\mu\eta_{\nu\sigma} - \delta^\varrho_\nu\eta_{\mu\sigma}
\end{aligned}
$$

Notice: viewing V^σ as a column vector, the index ϱ of $S_{\mu\nu}{}^\varrho{}_\sigma$ is a row index of $S_{\mu\nu}$, and σ is a column index. Hence, the matrix product $S_{\mu\nu}S_{\varrho\sigma}$ reads

$$
\begin{aligned}
(S_{\mu\nu}S_{\varrho\sigma})^\lambda{}_\tau &= S_{\mu\nu}{}^\lambda{}_\varphi S_{\varrho\sigma}{}^\varphi{}_\tau = (\delta^\lambda_\mu\eta_{\nu\varphi} - \delta^\lambda_\nu\eta_{\mu\varphi})(\delta^\varphi_\varrho\eta_{\sigma\tau} - \delta^\varphi_\sigma\eta_{\varrho\tau}) \\
&= \delta^\lambda_\mu\eta_{\nu\varrho}\eta_{\sigma\tau} - \delta^\lambda_\nu\eta_{\mu\varrho}\eta_{\sigma\tau} - \delta^\lambda_\mu\eta_{\nu\sigma}\eta_{\varrho\tau} + \delta^\lambda_\nu\eta_{\mu\sigma}\eta_{\varrho\tau} \\
\Rightarrow\quad [S_{\mu\nu}S_{\varrho\sigma}]^\lambda{}_\tau &= (S_{\mu\nu}S_{\varrho\sigma})^\lambda{}_\tau - (\mu\leftrightarrow\varrho, \nu\leftrightarrow\sigma) \\
&= \delta^\lambda_\mu\eta_{\nu\varrho}\eta_{\sigma\tau} - \delta^\lambda_\sigma\eta_{\varrho\nu}\eta_{\mu\tau} + \dots \\
&= \eta_{\nu\varrho}(\delta^\lambda_\mu\eta_{\sigma\tau} - \delta^\lambda_\sigma\eta_{\mu\tau}) + \dots = \eta_{\nu\varrho}S_{\mu\sigma}{}^\lambda{}_\tau + \dots \\
&= (\eta_{\nu\varrho}S_{\mu\sigma} - \eta_{\mu\varrho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\varrho} + \eta_{\mu\sigma}S_{\nu\varrho})^\lambda{}_\tau
\end{aligned}
$$

For S_{01} we obtain $S_{01}{}^\varrho{}_\sigma = \delta^\varrho_0\eta_{1\sigma} - \delta^\varrho_1\eta_{0\sigma}$ which shows that the only nonvanishing entries of S_{01} are $S_{01}{}^0{}_1 = 1$ and $S_{01}{}^1{}_0 = 1$. Analogously one determines S_{12}. In four dimensions this gives

$$
S_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Solution of exercise 12

a) For $\mu = 0$ we have $\Gamma_0\Gamma_0\Gamma_0 = -\Gamma_0 = \Gamma_0{}^\dagger$ owing to $\Gamma_0\Gamma_0 = -\mathbf{1}$ ($\Leftarrow$ Dirac algebra) and $-\Gamma_0 = \Gamma_0{}^{-1} = \Gamma_0{}^\dagger$ ($\Leftarrow$ Dirac algebra and unitarity of Γ_0). For $\mu = i$ we have $\Gamma_0\Gamma_i\Gamma_0 = -\Gamma_i\Gamma_0\Gamma_0 = \Gamma_i = \Gamma_i{}^{-1} = \Gamma_i{}^\dagger$ ($\Leftarrow$ Dirac algebra and unitarity of Γ_i).

b) $\Sigma_{\mu\nu}{}^\dagger = \frac{1}{4}(\Gamma_\mu\Gamma_\nu)^\dagger - (\mu\leftrightarrow\nu) = \frac{1}{4}(\Gamma_\nu{}^\dagger\Gamma_\mu{}^\dagger) - (\mu\leftrightarrow\nu) = \frac{1}{4}(\Gamma_0\Gamma_\nu\Gamma_0\Gamma_0\Gamma_\mu\Gamma_0) - (\mu\leftrightarrow\nu) = \frac{1}{4}(-\Gamma_0\Gamma_\nu\Gamma_\mu\Gamma_0) - (\mu\leftrightarrow\nu) = \frac{1}{4}\Gamma_0[\Gamma_\mu, \Gamma_\nu]\Gamma_0 = \Gamma_0\Sigma_{\mu\nu}\Gamma_0$.

c) $M^s_{\mu\nu}\psi^\dagger = M^s_{\mu\nu}\psi^{*\top} = (M^s_{\mu\nu}\psi^*)^\top = (M^s_{\mu\nu}\psi)^{*\top} = (-\Sigma_{\mu\nu}\psi)^\dagger = -\psi^\dagger\Sigma_{\mu\nu}{}^\dagger = -\psi^\dagger\Gamma_0\Sigma_{\mu\nu}\Gamma_0$.

d) $M^s_{\mu\nu}\bar\psi = M^s_{\mu\nu}(\psi^\dagger\Gamma_0) = (M^s_{\mu\nu}\psi^\dagger)\Gamma_0 = -\psi^\dagger\Gamma_0\Sigma_{\mu\nu}\Gamma_0\Gamma_0 = \psi^\dagger\Gamma_0\Sigma_{\mu\nu} = \bar\psi\Sigma_{\mu\nu}$.

Solution of exercise 13

$$
\begin{aligned}
&\frac{\partial L}{\partial\varphi_{,\mu}} = -\varphi_{,\nu}\eta^{\mu\nu} = -\partial^\mu\varphi \quad\Rightarrow\quad T_\nu{}^\mu = -\partial_\nu\varphi\partial^\mu\varphi + \delta^\mu_\nu(\frac{1}{2}\partial_\varrho\varphi\partial^\varrho\varphi + \frac{m^2}{2}\varphi^2) \\
&\Rightarrow\quad T_0{}^0 = -\frac{1}{2}\partial_0\varphi\partial^0\varphi + \frac{1}{2}\partial_i\varphi\partial^i\varphi + \frac{m^2}{2}\varphi^2 = \frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}\sum_i(\partial_i\varphi)^2 + \frac{m^2}{2}\varphi^2 \geq 0.
\end{aligned}
$$