2 Symmetries in classical field theory

2.1 Notation and correspondence to classical mechanics

The main complication of classical field theory as compared to classical mechanics is that the dynamical variables ('fields') depend on more than one variable. Fields ϕ^i may be understood as mappings from a base space $\mathcal{B}$ to a target space $\mathcal{Z}$,

$$\phi^i : \quad \mathcal{B} \to \mathcal{Z}, \quad (x^0, x^1, \dots, x^{n-1}) \mapsto \phi^i(x^0, x^1, \dots, x^{n-1}).$$

n denotes the dimension of the base space, coordinates of the base space are denoted by x^μ, $\mu = 0, 1, \dots, n-1$. Often the base space is $\mathcal{B} = \mathbb{R}^n$ and represents n-dimensional spacetime, with x^0 the time coordinate, and x^i, $i = 1, \dots, n-1$ the space coordinates. The target space depends on the field theory considered. E.g., a Dirac spinor field in 4-dimensional spacetime has 4 complex valued components, i.e., in this case the target space is $\mathcal{Z} = \mathbb{C}^4$.

The equations of motion of a field theory are partial differential equations rather than ordinary differential equations as in classical mechanics of point masses. Furthermore, in a field theory the conserved quantities which occur in Noethers first theorem are conserved currents J^μ (see below) rather than constants of motion as in classical mechanics. Apart from these differences, the description and theory of symmetries in classical field theory is a rather straightforward generalization of the one in classical mechanics. Actually, a classical mechanical system can be viewed as a field theory with a 1-dimensional base space.

The jet space approach can also be used in classical field theory: the jet variables are now the coordinates x^μ of the base space, the fields ϕ^i and jet variables $\phi^i_{,\mu}$, $\phi^i_{,\mu\nu}$, … which represent the first and higher order derivatives of the fields with respect to the coordinates x^μ.

The following table collects quantities in classical mechanics and their counterparts in field theory which are relevant to symmetries. In addition it introduces some notation.

	Mechanics	Field theory
base space coordinates	t	$x^\mu,\ \mu = 0, ..., n-1$
dynamical variables	$q^i,\ i = 1, ..., N$	$\phi^i,\ i = 1, ..., N$
derivatives (jet space)	$\dot{q}^i = \frac{dq^i}{dt},\ \ddot{q}^i = \frac{d^2q^i}{dt^2},\ ...$	$\phi^i_{,\mu} = \partial_\mu \phi^i,\ \phi^i_{,\mu\nu} = \partial_\nu \partial_\mu \phi^i,\ ...$
	$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + ...$	$\partial_\mu = \frac{\partial}{\partial x^\mu} + \phi^i_{,\mu} \frac{\partial}{\partial \phi^i} + \phi^i_{,\mu\nu} \frac{\partial}{\partial \phi^i_{,\nu}} + ...$
trajectories/mappings	$q^i(t)$,	$\phi^i(x) = \phi^i(x^0, ..., x^{n-1})$,
	$\dot{q}^i(t) = \frac{dq^i(t)}{dt},\ ...$	$\phi^i_{,\mu}(x) = \frac{\partial \phi^i(x)}{\partial x^\mu},\ ...$
action	$S[q] = \int dt\, L([q], t)$	$S[\phi] = \int d^n x\, L([\phi], x)$
		$d^n x = dx^0 ... dx^{n-1}$
	$L([q], t) = L(q^i, \dot{q}^i, ..., t)$	$L([\phi], x) = L(\phi^i, \phi^i_{,\mu}, ..., x^0, ..., x^{n-1})$
Euler-Lagr. operator	$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} + ...$	$\frac{\delta}{\delta \phi^i} = \frac{\partial}{\partial \phi^i} - \partial_\mu \frac{\partial}{\partial \phi^i_{,\mu}} + ...$
equations of motion	$\mathcal{L}_i([q], t) = 0$ (Lagr. systems: $\mathcal{L}_i = \frac{\delta L}{\delta q^i}$)	$\mathcal{L}_i([\phi], x) = 0$ (Lagr. systems: $\mathcal{L}_i = \frac{\delta L}{\delta \phi^i}$)
infinites. symmetry	$\delta_\varepsilon L([q], t) = \dot{K}([q], t)$	$\delta_\varepsilon L([\phi], x) = \partial_\mu K^\mu([\phi], x)$
conservation laws	$\dot{J}([q], t) = G^i \mathcal{L}_i([q], t)$,	$\partial_\mu J^\mu([\phi], x) = G^i \mathcal{L}_i([\phi], x)$,
	$G^i = g^{i(0)}([q], t) + g^{i(1)}([q], t) \frac{d}{dt} + ...$	$G^i = g^i([\phi], x) + g^{i\mu}([\phi], x) \partial_\mu + ...$

2.2 Conserved currents and charges, weak equality

As remarked above, the field theoretical analog of mechanical constants of motion are conserved currents $J^\mu([\phi],x)$ whose n-dimensional divergence $\partial_\mu J^\mu([\phi(x)],x)$ vanishes *for all solutions $\phi^i(x)$ of the equations of motion*. Therefore we denote current conservation by

$$\partial_\mu J^\mu([\phi],x) \approx 0 \tag{2.1}$$

where $\approx$ denotes weak equality defined according to

$$\begin{aligned} A([\phi],x) \approx B([\phi],x) \quad :\Leftrightarrow \quad & A([\phi],x) - B([\phi],x) = G^i \mathcal{L}_i([\phi],x), \\ & G^i = g^i([\phi],x) + g^{i\mu}([\phi],x)\partial_\mu + \ldots \end{aligned} \tag{2.2}$$

According to this definition two functions that are weakly equal coincide whenever evaluated for fields $\phi^i(x)$ which fulfill the equations of motion ($\mathcal{L}_i([\phi(x)],x) = 0$)[4].

(2.1) is a continuity equation: splitting up the sum $\partial_\mu J^\mu$ into the temporal part and the spatial part, it reads

$$\partial_0 J^0 + \mathrm{div}\vec{J} \approx 0, \tag{2.3}$$

where we have set $\vec{J} = (J^1, \ldots, J^{n-1})$ and div denotes the spatial divergence $\mathrm{div}\vec{J} = \partial_i J^i$.

Often one associates to J^μ a charge defined according to

$$Q = \int_V dV\, J^0([\phi],x), \tag{2.4}$$

where V is a fixed (time independent) $(n-1)$-dimensional spatial volume. Equation (2.3) implies

$$\begin{aligned} \dot{Q} &= \int_V dV\, \partial_0 J^0([\phi(x)],x) \\ &\approx -\int_V dV\, \mathrm{div}\vec{J}([\phi(x)],x) \\ &= -\int_{\partial V} d\vec{S}\, \vec{J}([\phi(x)],x), \end{aligned} \tag{2.5}$$

where we used the divergence theorem of vector analysis (Gauss-Ostrogradsky theorem) stating that the volume integral of a divergence $\mathrm{div}\vec{F}$ over V is equal to the surface integral of $\vec{F}$ over the boundary ∂V of V ($d\vec{S}$ denotes the surface elements of ∂V with surface normal pointing outward). For solutions $\phi^i(x)$ to the equations of motion that fall off sufficiently fast at the boundary ∂V, (2.5) implies $\dot{Q} = 0$. Hence, charges defined according to (2.4) in terms of conserved currents are conserved quantities when evaluated for solutions of the equations of motion that fall off sufficiently fast at the boundary of the integration volume V.

2.3 Global symmetries

Symmetries in field theory are defined analogously to symmetries in classical mechanics. We denote infinitesimal global symmetry transformations of the fields by

$$\delta_\varepsilon \phi^i = \varepsilon\, Q^i([\phi],x).$$

They arise from finite transformations

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu - \varepsilon\, \xi^\mu(x) + O(\varepsilon^2),$$

[4]Notice that (1.69) is just the special case of (2.1) for $n = 1$.

$$\phi^i(x) \mapsto \tilde{\phi}^i(\tilde{x}) = \phi^i(x) + \varepsilon\, \hat{Q}^i(x) + O(\varepsilon^2)$$

according to

$$\delta_\varepsilon \phi^i(x) = \tilde{\phi}^i(x) - \phi^i(x) + O(\varepsilon^2) = \varepsilon\Big(\xi^\mu(x)\partial_\mu\phi^i(x) + \hat{Q}^i(x)\Big) =: \varepsilon\, Q^i(x),$$

where we used the abbreviation $\xi^\mu(x) = \xi^\mu([\phi(x)], x)$ and analogously for $\hat{Q}^i(x)$ and $Q^i(x)$, as well as the Taylor expansion

$$\tilde{\phi}^i(\tilde{x}) = \tilde{\phi}^i(x - \varepsilon\xi + O(\varepsilon^2)) = \tilde{\phi}^i(x) - \varepsilon\xi^\mu(x)\partial_\mu\phi^i(x) + O(\varepsilon^2).$$

Analogously to the definition we used in classical mechanics, $\delta_\varepsilon\phi^i$ measures the change of the function $\phi^i(x)$ at fixed arguments $x^0, \dots, x^{n-1}$; infinitesimal transformations $\varepsilon\xi^\mu$ of the base space coordinates x^μ are contained in $\delta_\varepsilon\phi^i$ through the shift term $\varepsilon\xi^\mu\partial_\mu\phi^i$. Accordingly, δ_ε leaves the base space coordinates inert and commutes with the ∂_μ,

$$\delta_\varepsilon x^\mu = 0, \quad [\delta_\varepsilon, \partial_\mu] = 0.$$

In jet space, δ_ε takes the form

$$\delta_\varepsilon = \varepsilon\, Q^i([\phi], x)\frac{\partial}{\partial\phi^i} + \varepsilon\,(\partial_\mu Q^i([\phi], x))\frac{\partial}{\partial\phi^i_{,\mu}} + \dots$$

Finite and infinitesimal symmetries of an action $S[\phi] = \int d^n x\, L([\phi], x)$ are defined analgously to our definitions in classical mechanics, with total divergences $\partial_\mu K^\mu$ in place of total time derivatives $\dot{K}$. Hence, δ_ε is a symmetry of $S[\phi]$ if there are functions $K^\mu([\phi], x)$ such that

$$\delta_\varepsilon\, L([\phi], x) = \partial_\mu K^\mu([\phi], x).$$

2.4 Noethers first theorem (simple version)

The field theoretical analog of Noethers first theorem as presented in section 1.8 is straightforward. For notational simplicity we assume again that the Lagrangian contains at most first order derivatives of the fields, and only sketch the derivation since it is very similar to the one in section 1.8.

a) We assume first that δ_ε is a symmetry of $S[\phi] = \int d^n x\, L(\phi, \partial\phi, x)$, i.e.,

$$\delta_\varepsilon L = \partial_\mu K^\mu$$

for some K^μ. Owing to

$$\delta_\varepsilon\, L = (\delta_\varepsilon\phi^i)\,\frac{\delta L}{\delta\phi^i} + \partial_\mu\Big((\delta_\varepsilon\phi^i)\,\frac{\partial L}{\partial\phi^i_{,\mu}}\Big)$$

we obtain

$$\partial_\mu J^\mu = (\delta_\varepsilon\phi^i)\,\frac{\delta L}{\delta\phi^i}, \quad J^\mu = K^\mu - (\delta_\varepsilon\phi^i)\,\frac{\partial L}{\partial\phi^i_{,\mu}}. \tag{2.6}$$

Hence, J^μ is a conserved current (which may vanish).

b) Next we assume that J^μ is a conserved current of a system with action $S[\phi] = \int d^n x\, L(\phi, \partial\phi, x)$, i.e.,

$$\partial_\mu J^\mu = (g^i + g^{i\mu}\partial_\mu + \dots)\,\frac{\delta L}{\delta\phi^i}$$

for some functions $g^i([\phi],x)$, $g^{i\mu}([\phi],x)$, ... Steps analogous to those in part b) of section 1.8 yield

$$\partial_\mu K^\mu = \varepsilon Q^i \frac{\partial L}{\partial \phi^i} + \varepsilon(\partial_\mu Q^i)\frac{\partial L}{\partial \phi^i_{,\mu}}, \quad K^\mu = \varepsilon\Big(J^\mu - g^{i\mu}\frac{\delta L}{\delta \phi^i} + \dots + Q^i \frac{\partial L}{\partial \phi^i_{,\mu}}\Big), \quad Q^i = g^i - \partial_\mu g^{i\mu} + \dots \tag{2.7}$$

Hence, δ_ε with $\delta_\varepsilon \phi^i = \varepsilon Q^i$ is a symmetry of $S[\phi]$.

Again, a) and b) do not establish a bijective correspondence of symmetries and conserved currents. On the one hand the current J^μ in (2.6) may vanish (cf. the example in the comments in section 1.8). On the other hand the Q^i in (2.7) can vanish because there are conserved currents whose divergence $\partial_\mu J^\mu$ vanishes identically (these currents are of the form $J^\mu = \partial_\nu K^{\nu\mu}$ with $K^{\nu\mu} = -K^{\mu\nu}$ and thus fulfill $\partial_\mu J^\mu = \partial_\mu \partial_\nu K^{\nu\mu} = 0$). A bijective correspondence of symmetries and conserved currents is obtained for equivalence classes of symmetries and conserved currents.

2.5 Energy and momentum conservation

As an application of Noethers first theorem in field theory we treat Lagrangians which do not depend explicitly on the base space coordinates x^μ,

$$\frac{\partial L}{\partial x^\mu} = 0 \quad \forall x^\mu.$$

In this case the transformations $\delta_\varepsilon \phi^i = \varepsilon^\mu \phi^i_{,\mu}$ constitute a symmetry of the action for every choice of constants ε^μ. Indeed,

$$\delta_\varepsilon \phi^i = \varepsilon^\mu \phi^i_{,\mu} \quad \Rightarrow \quad \delta_\varepsilon = \varepsilon^\mu\Big(\partial_\mu - \frac{\partial}{\partial x^\mu}\Big) \quad \Rightarrow \quad \delta_\varepsilon L = \partial_\mu(\varepsilon^\mu L).$$

For Lagrangians depending at most on first order derivatives of the fields, (2.6) yields the conserved currents

$$J^\mu = \varepsilon^\mu L - \varepsilon^\nu \phi^i_{,\nu}\frac{\partial L}{\partial \phi^i_{,\mu}} = -\varepsilon^\nu T_\nu{}^\mu, \quad T_\nu{}^\mu = \phi^i_{,\nu}\frac{\partial L}{\partial \phi^i_{,\mu}} - \delta^\mu_\nu L$$

This yields for each value of ν a conserved current $T_\nu{}^\mu$. In a relativistic theory the $T_\nu{}^\mu$ are the components of the energy-momentum tensor (field).

2.6 Technical remark on jet space operations

The jet variables $\phi^i_{,\mu\nu}$, $\phi^i_{,\mu\nu\varrho}$, ... which represent second or higher order derivatives of the fields are not all independent because they are symmetric in the derivative indices ($\phi^i_{,\mu\nu} = \phi^i_{,\nu\mu}$ etc.). This reflects that the partial derivatives represented by these indices commute (assuming well-behaved fields) and implies, e.g., that the jet variables $\phi^i_{,01}$ and $\phi^i_{,10}$ are to be considered identical. Working with an overcomplete set of variables is a subtle matter and can cause inconsistencies when not dealt with carefully. E.g.: Does the derivative of $\phi^i_{,10}$ w.r.t. $\phi^i_{,01}$ vanish, or not? Should both $\frac{\partial}{\partial \phi^i_{,01}}$ and $\frac{\partial}{\partial \phi^i_{,10}}$ occur in the Euler-Lagrange operator $\frac{\delta}{\delta \phi^i}$?

The problem can be handled in various ways. An obvious solution is to work with an independent set of jet variables, such as $\{\phi^i_{,\mu_1\dots\mu_k} : \mu_i \leq \mu_{i+1}, k = 0,1,\dots\}$ which contains $\phi^i_{,01}$ but not $\phi^i_{,10}$. The latter solution is clean but has several disadvantages, such as making formulae rather unpleasant and loosing manifest covariance in Poincaré or diffeomorphism invariant theories.

A more convenient way to handle the problem is the use of symmetrical derivative operations $\partial^S/\partial\phi^i_{,\mu_1\ldots\mu_k}$ defined according to

$$\frac{\partial^S\phi^j_{,\nu_1\ldots\nu_l}}{\partial\phi^i_{,\mu_1\ldots\mu_k}} = \delta^j_i\,\delta^k_l\,\delta^{\mu_1}_{(\nu_1}\ldots\delta^{\mu_k}_{\nu_k)}$$

where $(\nu_1\ldots\nu_k)$ denotes complete symmetrization with weight one, e.g.

$$\delta^{\mu_1}_{(\nu_1}\delta^{\mu_2}_{\nu_2)} = \frac{1}{2}\,(\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}).$$

This has unfamiliar consequences, such as

$$\frac{\partial^S\phi^1_{,01}}{\partial\phi^1_{,01}} = \frac{\partial^S\phi^1_{,01}}{\partial\phi^1_{,10}} = \frac{1}{2}\ ,$$

but avoids several unpleasant features of other approaches (such as complicated combinatorical factors in various formulae). In particular the Euler-Lagrange and the derivative operators take the simple forms

$$\frac{\delta}{\delta\phi^i} = \frac{\partial}{\partial\phi^i} - \partial_\mu\,\frac{\partial}{\partial\phi^i_{,\mu}} + \partial_\nu\partial_\mu\,\frac{\partial^S}{\partial\phi^i_{,\mu\nu}} + \cdots + (-)^k\partial_{\mu_1}\ldots\partial_{\mu_k}\,\frac{\partial^S}{\partial\phi^i_{,\mu_1\ldots\mu_k}} + \ldots$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \phi^i_{,\mu}\frac{\partial}{\partial\phi^i} + \phi^i_{,\nu\mu}\,\frac{\partial}{\partial\phi^i_{,\nu}} + \phi^i_{,\varrho\nu\mu}\,\frac{\partial^S}{\partial\phi^i_{,\varrho\nu}} + \cdots + \phi^i_{,\nu_1\ldots\nu_k\mu}\,\frac{\partial^S}{\partial\phi^i_{,\nu_1\ldots\nu_k}} + \ldots$$