1.3 Formulation and definition of continuous global symmetries

1.3.1 Infinitesimal symmetries

We shall use the following general structure and mathematical set-up of infinitesimal global symmetry transformations δ_ε:

$$\delta_\varepsilon q^i(t) = \varepsilon\, Q^i([q],t), \quad \varepsilon = constant \tag{1.11}$$

$$\delta_\varepsilon t = 0 \tag{1.12}$$

$$\left[\delta_\varepsilon, \frac{d}{dt}\right] = 0 \tag{1.13}$$

$$\delta_\varepsilon(f\,g) = (\delta_\varepsilon f)\,g + f\,(\delta_\varepsilon g) \quad \forall f,g \quad \Big(\text{with } f = f([q],t)),\ g = g([q],t)\Big) \tag{1.14}$$

Comments:

- (1.11) is the general form of an infinitesimal symmetry transformation of q^i. ε is a constant small parameter (it may be omitted or absorbed in Q^i, but I prefer to write it explicitly for later purposes). The notation $Q^i([q],t)$ means that Q^i can depend on $q^i, \dot q^i, \ddot q^i, \dots$ and also explicitly on t. Of course, since δ_ε is to be a symmetry, the Q^i are not arbitrary functions but are subject to Eq. (1.16) below.
- (1.12) states that t is not transformed. As we shall see later, this can always be imposed (with no loss of generality), even in cases such as 'time translation symmetry'. The reason is that transformations of t can be 'shifted' to transformations of the q^i, as we shall discuss later.
- According to (1.13), infinitesimal variations commute with the total time derivative. On q^i this reflects that $\delta_\varepsilon \dot q^i$ is the total time derivative of $\delta_\varepsilon q^i$, and analogously for the transformations of higher order time derivatives of q^i,

  $$\delta_\varepsilon \dot q^i = \varepsilon\, \dot Q^i([q],t), \quad \delta_\varepsilon \ddot q^i = \varepsilon\, \ddot Q^i([q],t), \dots$$

  On t, (1.13) holds owing to (1.12).
- (1.14) states that δ_ε is a so-called derivation, i.e. it fulfills the product rule (on functions of the $q^i, \dot q^i, \ddot q^i, \dots, t$).
- A transformation of this type is called 'global' because of the constancy of the parameter ε. Later we shall meet 'local' transformations whose parameters are arbitrary functions of t rather than constants.
- Equations (1.11)–(1.14) can be summarized by writing δ_ε as a differential operator:

  $$\delta_\varepsilon = \varepsilon\, Q^i([q],t)\frac{\partial}{\partial q^i} + \varepsilon\, \dot Q^i([q],t)\frac{\partial}{\partial \dot q^i} + \varepsilon\, \ddot Q^i([q],t)\frac{\partial}{\partial \ddot q^i} + \dots \tag{1.15}$$

  This differential operator acts on functions of the $q^i, \dot q^i, \ddot q^i, \dots, t$. Actually, in order to make this approach to symmetries mathematically precise, it can and should be formulated in the space of functions of the $q^i, \dot q^i, \ddot q^i, \dots, t$ (the mathematical term for this space is 'jet space'). Even though we have not introduced this mathematical setting formally, we have been using it implicitly and intuitively already (e.g., by viewing the Lagrangian as a function of the $q^i, \dot q^i, \ddot q^i, \dots, t$, and thus as a jet space function).

Definition: A derivation δ_ε as above is called an infinitesimal (global) symmetry of an action $S = \int dt\, L([q],t)$ if $\delta_\varepsilon L$ is a total time derivative, i.e., if there is a $K([q],t)$ such that

$$\delta_\varepsilon L([q],t) = \frac{d}{dt}\, K([q],t). \tag{1.16}$$

Comments:

- Owing to (1.8), equation (1.16) is equivalent to

$$\frac{\delta}{\delta q^i}\Big(\delta_\varepsilon L([q],t)\Big) = 0 \quad \forall q^i \tag{1.17}$$

- The occurrence of a total time derivative on the right hand side in (1.16) is motivated, among other things, by:
  - Two Lagrangians which differ only by a total time derivative are to be considered equivalent, at least as far as the equations of motions are concerned (because they give rise to the same equations of motions, cf. (1.7)). Hence, it is natural to define symmetries so that the symmetries of equivalent Lagrangians agree too. This is guaranteed by (1.16), as one has (as a consequence of (1.14)),

$$L_2 = L_1 + \frac{df}{dt} \quad \Rightarrow \quad \delta_\varepsilon L_2 = \delta_\varepsilon L_1 + \frac{d(\delta_\varepsilon f)}{dt} \tag{1.18}$$

  - As we shall see later, Noethers first theorem (stating the correspondence of global symmetries and conservations laws) holds in its general form only for symmetries defined according to (1.16).
- Notice that (1.16) implies, in general, invariance of the action $S = \int dt\, L$ under δ_ε only up to a 'surface term' (which does not vanish in general):

$$\begin{aligned}\delta_\varepsilon L = \frac{dK}{dt} \quad \Rightarrow \quad \delta_\varepsilon S &= \int_{t_1}^{t_2} dt\, \delta_\varepsilon L = \int_{t_1}^{t_2} dt\, \frac{dK}{dt}\\ &= K\Big|_{t_1}^{t_2} = K([q(t_2)],t_2) - K([q(t_1)],t_1)\end{aligned} \tag{1.19}$$

Examples:

1. Free 1-dimensional motion:

$$L = \frac{m}{2}\,\dot x^2 \tag{1.20}$$

$$\delta_\varepsilon x = \varepsilon \quad \Rightarrow \quad \delta_\varepsilon L = 0 \tag{1.21}$$

$$\delta_\eta x = \eta \dot x \quad \Rightarrow \quad \delta_\eta L = \eta\, m\, \ddot x\, \dot x = \frac{d}{dt}\Big(\eta\, \frac{m}{2}\, \dot x^2\Big) \tag{1.22}$$

2. **Exercise 2:** Verify that δ_ε is a symmetry of L with

$$L = \frac{m}{2}\,(\dot x^2 + \dot y^2 + \dot z^2) - U(r), \quad r = \sqrt{x^2+y^2+z^2} \tag{1.23}$$

$$\delta_\varepsilon x = \varepsilon y, \quad \delta_\varepsilon y = -\varepsilon x, \quad \delta_\varepsilon z = 0 \quad \text{(infinites. rotation)} \tag{1.24}$$

3. **Exercise 3:** As exercise 2 for[1]

$$\begin{aligned}L([x,\varphi]) =& \frac{M+m}{2}\,\dot x^2\,(1+\tan^2\alpha) + \frac{m}{2}\,L^2\,\dot\varphi^2\\ &+ m\,L\,\dot\varphi\,\dot x\,(\cos\varphi + \tan\alpha\sin\varphi)\\ &- (M+m)\,g\,x\tan\alpha + m\,g\,L\,\cos\varphi\\ \delta_\varepsilon x = \varepsilon, \quad & \delta_\varepsilon\varphi = 0\end{aligned}$$

[1]Notice: this is the Lagrangian (1.10) derived in exercise 1 (section 1.2).

Solution to exercise 2:

$$\begin{aligned}
\delta_\varepsilon L =&(\delta_\varepsilon x)\,\frac{\partial L}{\partial x}+(\delta_\varepsilon \dot{x})\,\frac{\partial L}{\partial \dot{x}}+\text{analogous terms for } y \text{ and } z\\
=&(\delta_\varepsilon x)\,\frac{dU}{dr}\,\frac{\partial r}{\partial x}+(\delta_\varepsilon \dot{x})\,\frac{\partial L}{\partial \dot{x}}+\text{analogous terms for } y \text{ and } z\\
=&(\delta_\varepsilon x)\,\frac{dU}{dr}\,\frac{x}{r}+(\delta_\varepsilon \dot{x})\,\frac{\partial L}{\partial \dot{x}}+\text{analogous terms for } y \text{ and } z\\
=&\varepsilon y\,\frac{dU}{dr}\,\frac{x}{r}+\varepsilon\dot{y}\,\frac{\partial L}{\partial \dot{x}}-\varepsilon x\,\frac{dU}{dr}\,\frac{y}{r}-\varepsilon\dot{x}\,\frac{\partial L}{\partial \dot{y}}\\
=&\varepsilon y\,\frac{dU}{dr}\,\frac{x}{r}+\varepsilon\dot{y}\,m\,\dot{x}-\varepsilon x\,\frac{dU}{dr}\,\frac{y}{r}-\varepsilon\dot{x}\,m\,\dot{y}\\
=&0
\end{aligned}$$

Solution to exercise 3:

$$\begin{aligned}
\delta_\varepsilon L =&(\delta_\varepsilon x)\,\frac{\partial L}{\partial x}+(\delta_\varepsilon \dot{x})\,\frac{\partial L}{\partial \dot{x}}+(\delta_\varepsilon \varphi)\,\frac{\partial L}{\partial \varphi}+(\delta_\varepsilon \dot{\varphi})\,\frac{\partial L}{\partial \dot{\varphi}}\\
=&\varepsilon\,\frac{\partial L}{\partial x}=-\varepsilon\,(M+m)\,g\,\tan\alpha=\frac{d}{dt}\Big(-t\,\varepsilon\,(M+m)\,g\,\tan\alpha\Big)
\end{aligned} \tag{1.25}$$

1.3.2 Finite symmetries

Definition: A transformation $t \mapsto \tilde{t}(t)$, $q^i(t) \mapsto \tilde{q}^i(\tilde{t})$ is called a symmetry of $S[q] = \int dt\, L([q(t)], t)$ if $S[\tilde{q}] = \int d\tilde{t}\, L([\tilde{q}(\tilde{t})], \tilde{t})$ equals $S[q]$, except for an integrated total time derivative $\int dt\, \frac{dK([q(t)],t)}{dt}$ (K may vanish), for *all* intervals of integration and *all* trajectories $q^i(t)$,

$$\int_{\tilde{t}_1(t_1)}^{\tilde{t}_2(t_2)} d\tilde{t}\, L([\tilde{q}(\tilde{t})], \tilde{t}) = \int_{t_1}^{t_2} dt \Big(L([q(t)], t) + \frac{dK([q(t)], t)}{dt} \Big) \quad \forall t_1, t_2, q^i(t). \tag{1.26}$$

Comment: Notice that $S[\tilde{q}]$ involves the *same* Lagrangian as $S[q]$: there is the *same* function L on the left hand side of (1.26) as on the right hand side (and *not* a function $\tilde{L}$ on the left hand side which may differ from the function L on the right hand side; just the arguments of L on the left hand side differ from those on the right hand side).

Examples: We shall now present five different symmetries of the following action describing a point mass m in a homogeneous gravitational field of acceleration g:

$$S[x, y, z] = m \int dt \Big(\tfrac{1}{2} \left(\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t) \right) - g\, z(t) \Big) \tag{1.27}$$

Each symmetry involves a constant parameter α (we consider each symmetry separately from the others and use the same symbol for the respective parameter).

- Constant shifts of x:
$$\tilde{t} = t, \quad \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \begin{pmatrix} x(t) + \alpha \\ y(t) \\ z(t) \end{pmatrix} \tag{1.28}$$
This is a symmetry of (1.27) with $S[\tilde{x}, \tilde{y}, \tilde{z}] = S[x, y, z]$.
- Constant shifts of y:
$$\tilde{t} = t, \quad \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) + \alpha \\ z(t) \end{pmatrix} \tag{1.29}$$
This is a symmetry of (1.27) with $S[\tilde{x}, \tilde{y}, \tilde{z}] = S[x, y, z]$.
- Constant shifts of z:
$$\tilde{t} = t, \quad \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) + \alpha \end{pmatrix} \tag{1.30}$$
This is a symmetry of (1.27) with
$$S[\tilde{x}, \tilde{y}, \tilde{z}] = S[x, y, z] + \int dt\, \frac{d(-m\, g\, \alpha\, t)}{dt}\,. \tag{1.31}$$
- Rotations around the z-axis:
$$\tilde{t} = t, \quad \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{1.32}$$
This is a symmetry of (1.27) with $S[\tilde{x}, \tilde{y}, \tilde{z}] = S[x, y, z]$.
- Constant shifts of t:
$$\tilde{t} = t - \alpha, \quad \begin{pmatrix} \tilde{x}(\tilde{t}) \\ \tilde{y}(\tilde{t}) \\ \tilde{z}(\tilde{t}) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{1.33}$$
This is a symmetry of (1.27) with $S[\tilde{x}, \tilde{y}, \tilde{z}] = S[x, y, z]$ (cf. exercise 4 below).

Exercise 4: Consider the following action (free 1-dimensional motion of a point mass m) and transformations

$$S[x] = \frac{m}{2}\int dt\,\dot{x}^2(t), \quad \tilde{t} = h(t), \quad \tilde{x}(\tilde{t}) = x(t). \tag{1.34}$$

Determine all functions $h(t)$ for which $t \mapsto \tilde{t}$, $x(t) \mapsto \tilde{x}(\tilde{t})$ is a symmetry of $S[x]$.

Solution of exercise 4:

$$S[\tilde{x}] = \int_{\tilde{t}_1(t_1)}^{\tilde{t}_2(t_2)} d\tilde{t}\, L([\tilde{x}(\tilde{t})]) = \frac{m}{2}\int_{\tilde{t}_1(t_1)}^{\tilde{t}_2(t_2)} d\tilde{t}\,\Big(\frac{d\tilde{x}(\tilde{t})}{d\tilde{t}}\Big)^2. \tag{1.35}$$

In order to compare this expression to $S[x]$, we switch the integration variable from $\tilde{t}$ to t, using

$$d\tilde{t} = dt\,\frac{d\tilde{t}}{dt} = dt\,\frac{dh(t)}{dt} = dt\,\dot{h}(t), \tag{1.36}$$

$$\frac{d\tilde{x}(\tilde{t})}{d\tilde{t}} = \frac{dt}{d\tilde{t}}\,\frac{dx(t)}{dt} = \frac{1}{\dot{h}(t)}\,\dot{x}(t) \tag{1.37}$$

(Notice: in (1.37) we have used $\tilde{x}(\tilde{t}) = x(t)$). This gives

$$\frac{m}{2}\int_{\tilde{t}_1(t_1)}^{\tilde{t}_2(t_2)} d\tilde{t}\,\Big(\frac{d\tilde{x}(\tilde{t})}{d\tilde{t}}\Big)^2 = \frac{m}{2}\int_{t_1}^{t_2} dt\,\frac{1}{\dot{h}(t)}\,\dot{x}^2(t) \tag{1.38}$$

The requirement that this is equal to $\frac{m}{2}\int_{t_1}^{t_2} dt\,\dot{x}^2(t)$, except for an integrated total time derivative, for all intervals $[t_1, t_2]$ imposes

$$\frac{m}{2}\,\frac{1}{\dot{h}(t)}\,\dot{x}^2(t) \stackrel{!}{=} \frac{m}{2}\,\dot{x}^2(t) + \frac{dK([x(t)],t)}{dt}\,. \tag{1.39}$$

(1.39) must be satisfied for all trajectories $x(t)$. This means it translates to the equation

$$\frac{m}{2}\,\frac{1}{\dot{h}(t)}\,\dot{x}^2 \stackrel{!}{=} \frac{m}{2}\,\dot{x}^2 + \frac{dK([x],t)}{dt}\,. \tag{1.40}$$

which has to hold identically in the variables t, x and $\dot{x}$ (i.e., in jet space). Applying the Euler-Lagrange derivative with respect to x to (1.40), we obtain[2]

$$-\frac{d}{dt}\Big(\frac{1}{\dot{h}(t)}\,m\,\dot{x}\Big) \stackrel{!}{=} -m\,\ddot{x} \quad\Leftrightarrow\quad \Big(\frac{d}{dt}\,\frac{1}{\dot{h}(t)}\Big)\dot{x} + \frac{1}{\dot{h}(t)}\,\ddot{x} \stackrel{!}{=} \ddot{x}. \tag{1.41}$$

Since (1.41) must hold identically in $\dot{x}$ and $\ddot{x}$, we can compare the coefficient functions of $\dot{x}$ and $\ddot{x}$ separately:

$$\frac{d}{dt}\,\frac{1}{\dot{h}(t)} \stackrel{!}{=} 0 \quad\wedge\quad \frac{1}{\dot{h}(t)} \stackrel{!}{=} 1 \tag{1.42}$$

This imposes $\dot{h}(t) \stackrel{!}{=} 1$ and thus

$$h(t) = t - \alpha, \quad \alpha = constant. \tag{1.43}$$

[2]Recall that the total time derivative of any function has vanishing Euler-Lagrange derivatives. Hence, whatever function K in (1.40) is, the Euler-Lagrange derivative of $\frac{dK}{dt}$ w.r.t. x vanishes.

1.4 From finite to infinitesimal global symmetries

We consider finite symmetries involving a constant parameter whose values can vary continuously over some interval such that for one of these values the symmetry transformations reduce to the identity. In the following we shall denote this parameter by α and assume, with no loss of generality, that $\alpha = 0$ gives the identity. Examples can be found in section 1.3.2 (in all these examples α can take all values between $-\infty$ and $+\infty$). Such symmetries are called global symmetries continuously connected to the identity: global because α does not depend on t, continuous because α can vary continuously over a certain range, connected to the identity because there is a value of α which reproduces the identity. In the next few lines we shall emphasize the presence of α by denoting finite symmetry transformations of this type by

$$t \mapsto \tilde{t}(t;\alpha), \quad q^i(t) \mapsto \tilde{q}^i(\tilde{t};\alpha) \quad \text{with} \quad \tilde{t}(t;0) = t, \quad \tilde{q}^i(\tilde{t}(t;0);0) = q^i(t). \tag{1.44}$$

Furthermore we assume that the transformations can be expanded as power series' in α around $\alpha = 0$. The infinitesimal version of such symmetry transformations (along the lines of section 1.3.1) is defined according to

$$\delta_\varepsilon t = 0 \tag{1.45}$$

$$\begin{aligned}\delta_\varepsilon q^i(t) &= \tilde{q}^i(t;\varepsilon) - q^i(t) \text{ to first order in } \varepsilon \\ &= \varepsilon \lim_{\alpha\to 0} \frac{\tilde{q}^i(t;\alpha) - q^i(t;0)}{\alpha} \\ &= \varepsilon \frac{\partial \tilde{q}^i(t;\alpha)}{\partial\alpha}\Big|_{\alpha=0}\end{aligned} \tag{1.46}$$

Notice that, as in section 1.3.1, δ_ε vanishes on t, see (1.45), even though in (1.44) we allow transformations $t \mapsto \tilde{t}$ with $\tilde{t} \neq t$. Furthermore, according to (1.46), $\delta_\varepsilon q^i(t)$ compares $\tilde{q}^i$ and q^i at the *same* argument t (rather than $\tilde{q}^i(\tilde{t})$ and $q^i(t)$). This means that $\delta_\varepsilon q^i(t)$ measures the infinitesimal change of the *function* $q^i(t)$.

In order to explain that and why this definition of infinitesimal transformations makes sense, we shall prove:

Lemma: If a set of transformations (1.44) constitutes a symmetry of an action $S[q] = \int dt\, L([q(t)], t)$, then the corresponding transformations δ_ε defined according to (1.45) and (1.46) constitute an infinitesimal symmetry of $S[q]$.

Proof: Expanding $\tilde{t}$ and $\tilde{q}^i$ in ε and dropping the argument ε in $\tilde{t}$ and $\tilde{q}^i$ (i.e., using $\tilde{t}(t)$ in place of $\tilde{t}(t;\varepsilon)$ etc.), we have

$$\tilde{t}(t) = t - \varepsilon\, f(t) + O(\varepsilon^2) \tag{1.47}$$

$$\tilde{q}^i(\tilde{t}) = q^i(t) + \varepsilon\, \hat{Q}^i(t) + O(\varepsilon^2) \tag{1.48}$$

$$\Rightarrow \quad \tilde{q}^i(t - \varepsilon\, f(t)) + O(\varepsilon^2) = \tilde{q}^i(t) - \varepsilon\, f(t)\, \dot{q}^i(t) + O(\varepsilon^2) = q^i(t) + \varepsilon\, \hat{Q}^i(t) + O(\varepsilon^2)$$

$$\Rightarrow \quad \delta_\varepsilon\, q^i(t) = \tilde{q}^i(t) - q^i(t) + O(\varepsilon^2) = \varepsilon\, f(t)\, \dot{q}^i(t) + \varepsilon\, \hat{Q}^i(t). \tag{1.49}$$

for some functions $f(t)$ and $\hat{Q}(t)$ (we have used the Taylor expansion $\tilde{q}^i(t - \varepsilon f(t)) = \tilde{q}^i(t) - \varepsilon f(t)\, \dot{\tilde{q}}^i(t) + O(\varepsilon^2) = \tilde{q}^i(t) - \varepsilon f(t)\, \dot{q}^i(t) + O(\varepsilon^2)$). We now examine $S[\tilde{q}]$ to first order in ε:

$$\begin{aligned}S[\tilde{q}] &= \int_{\tilde{t}_1(t_1)}^{\tilde{t}_2(t_2)} d\tilde{t}\, L([\tilde{q}(\tilde{t})], \tilde{t}) \\ &= \int_{t_1}^{t_2} dt\, \Big(1 - \varepsilon\, \dot{f}(t)\Big)\, L([\tilde{q}(t - \varepsilon\, f(t))], t - \varepsilon\, f(t)) + O(\varepsilon^2)\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_1}^{t_2} dt\,\Big(1 - \varepsilon\,\dot f(t)\Big)\Big(L([\tilde q(t)],t) - \varepsilon\,f(t)\,\dot L([q(t)],t)\Big) + O(\varepsilon^2)\\
&= \int_{t_1}^{t_2} dt\,\Big(1 - \varepsilon\,\dot f(t)\Big)\Big(L([q(t)+\delta_\varepsilon q(t)],t) - \varepsilon\,f(t)\,\dot L([q(t)],t)\Big) + O(\varepsilon^2)\\
&= \int_{t_1}^{t_2} dt\,\Big(L([q(t)+\delta_\varepsilon q(t)],t) - \varepsilon\,\frac{d}{dt}\Big(f(t)\,L([q(t)],t)\Big)\Big) + O(\varepsilon^2), \qquad (1.50)
\end{aligned}
$$

where from line 1 to 2 we switched the variable of integration from $\tilde t$ to t (the factor $1-\varepsilon\dot f(t)$ results from $d\tilde t = d(t-\varepsilon f(t)) = dt\,(1-\varepsilon\dot f(t))$, from line 2 to 3 we used the Taylor expansion $L([\tilde q(t-\varepsilon f)], t-\varepsilon f) = L([\tilde q(t)],t) - \varepsilon f\dot L([q(t)],t) + O(\varepsilon^2)$, from line 3 to 4 we used $\tilde q(t) = q(t)+\delta_\varepsilon q(t)$, cf. (1.49), and from line 4 to 5 we rearranged terms. Now, by assumption, $t\mapsto \tilde t(t)$, $q^i(t)\mapsto \tilde q^i(\tilde t)$ is a symmetry of $S[q]$, i.e.,

$$
\int_{\tilde t_1(t_1)}^{\tilde t_2(t_2)} d\tilde t\,L([\tilde q(\tilde t)],\tilde t) - \int_{t_1}^{t_2} dt\,L([q(t)],t) = \int_{t_1}^{t_2} dt\,\frac{dK([q(t)],t)}{dt} \qquad (1.51)
$$

for some $K([q(t)],t)$. Using (1.50) in (1.51), we obtain

$$
\begin{aligned}
\int_{t_1}^{t_2} dt\,\Big(L([q(t)+\delta_\varepsilon q(t)],t) - \varepsilon\,\frac{d}{dt}\Big(f(t)\,L([q(t)],t)\Big) - L([q(t)],t)\Big)&\\
= \int_{t_1}^{t_2} dt\,\varepsilon\,\frac{d\hat K([q(t)],t)}{dt} + O(\varepsilon^2),& \qquad (1.52)
\end{aligned}
$$

where we used that $K([q(t)],t) = \varepsilon\hat K([q(t)],t) + O(\varepsilon^2)$ for some $\hat K$ (K vanishes at $\varepsilon = 0$ since $S[\tilde q]$ equals $S[q]$ for $\varepsilon = 0$). Rearranging (1.52), it reads

$$
\begin{aligned}
&\int_{t_1}^{t_2} dt\,\Big(L([q(t)+\delta_\varepsilon q(t)],t) - L([q(t)],t)\Big)\\
&= \int_{t_1}^{t_2} dt\,\varepsilon\,\frac{d}{dt}\Big(\hat K([q(t)],t) + f(t)\,L([q(t)],t)\Big) + O(\varepsilon^2). \qquad (1.53)
\end{aligned}
$$

Notice that the integrand on the left hand side is just $\delta_\varepsilon L + O(\varepsilon^2)$,

$$
L([q(t)+\delta_\varepsilon q(t)],t) - L([q(t)],t) = \delta_\varepsilon L([q(t)],t) + O(\varepsilon^2). \qquad (1.54)
$$

Furthermore, recall that (1.53) holds for all intervals $[t_1,t_2]$. This implies (to $\mathrm{O}(\varepsilon)$):

$$
\delta_\varepsilon L([q(t)],t) = \varepsilon\,\frac{d}{dt}\Big(\hat K([q(t)],t) + f(t)\,L([q(t)],t)\Big). \qquad (1.55)
$$

Since (1.55) holds for all trajectories $q^i(t)$, it gives rise to an equation in jet space. In jet space $f(t)$ has to be replaced by $f([q],t)$, for in general f can also involve the q^i and time derivatives thereof. Hence, in jet space (1.55) becomes

$$
\delta_\varepsilon L([q],t) = \varepsilon\,\frac{d}{dt}\Big(\hat K([q],t) + f([q],t)\,L([q],t)\Big). \qquad (1.56)
$$

This completes the proof since (1.56) shows that δ_ε is indeed an infinitesimal symmetry of $S[q]$ in accordance with the definition in section 1.3.1.

Comment: The proof shows that transformations $\tilde t(t) = t - \varepsilon f(t)$ can be absorbed into the transformations of the q^i through contributions $\varepsilon f(t)\,\dot q^i(t)$ to $\delta_\varepsilon q^i(t)$. As a consequence, there is a term $\varepsilon\frac{d}{dt}(f(t)\,L([q],t))$ in $\delta_\varepsilon L([q],t)$.

1.5 From infinitesimal to finite global symmetries

Finite symmetry transformations are obtained from the infinitesimal symmetry transformations δ_ε according to

$$\tilde{t} = t, \quad \tilde{q}^i = \exp(\delta_\varepsilon)\, q^i \tag{1.57}$$

(with $\tilde{q}^i(t) = (\exp(\delta_\varepsilon)q^i)(t)$) where $\exp(\delta_\varepsilon)$ is the exponential function of δ_ε:

$$\exp(\delta_\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^n. \tag{1.58}$$

Hence

$$\tilde{q}^i = \Big(1 + \delta_\varepsilon + \tfrac{1}{2}(\delta_\varepsilon)^2 + \tfrac{1}{6}(\delta_\varepsilon)^3 + \dots\Big) q^i$$

This interrelation of infinitesimal and finite transformations is based on the special features of the exponential function which imply that $\exp(\delta_\varepsilon)$ applied to any function $f([q],t)$ equals $f([\tilde{q}],t)$ with $\tilde{q}^i$ as in (1.57):

$$\exp(\delta_\varepsilon)\, f([q],t) = f([\exp(\delta_\varepsilon)q],t) \quad \forall f([q],t). \tag{1.59}$$

(For polynomials f and formal power series' this will be proven in an exercise). (1.59) implies the following result:

Lemma: If δ_ε is an infinitesimal symmetry of an action $S[q] = \int dt\, L([q],t)$, then the corresponding transformations (1.57) constitute a finite symmetry of $S[q]$.

Proof: Let δ_ε be an infinitesimal symmetry of $S[q] = \int dt\, L([q],t)$, i.e.,

$$\delta_\varepsilon\, L([q],t) = \varepsilon\, \frac{d}{dt}\, K([q],t) \tag{1.60}$$

for some function K (owing to $\tilde{t} = t$ we suppress the argument t of q^i and $\tilde{q}^i$). Then

$$\begin{aligned}
L([\tilde{q}],t) &\overset{(1.57)}{=} L([\exp(\delta_\varepsilon)q],t) \overset{(1.59)}{=} \exp(\delta_\varepsilon)\, L([q],t)\\
&\overset{(1.58)}{=} L([q],t) + \sum_{n=1}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^n L([q],t) = L([q],t) + \sum_{n=1}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^{n-1}\delta_\varepsilon L([q],t)\\
&\overset{(1.60)}{=} L([q],t) + \sum_{n=1}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^{n-1}\varepsilon\, \frac{d}{dt}\, K([q],t)\\
&\overset{(1.13)}{=} L([q],t) + \frac{d}{dt} \sum_{n=1}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^{n-1}\varepsilon\, K([q],t)\\
&= \ L([q],t) + \frac{d}{dt}\Big(\varepsilon\, \frac{\exp(\delta_\varepsilon) - 1}{\delta_\varepsilon}\, K([q],t)\Big).
\end{aligned} \tag{1.61}$$

Differential equations for $\tilde{q}^i$: Let us consider, for simplicity, infinitesimal transformations $\delta_\varepsilon q^i = \varepsilon Q^i(q,t)$ (where the Q^i depend only on the q^i and possibly explicitly on t but not on $\dot{q}^i$, $\ddot{q}^i$, $\dots$; the formulae can be generalized). Applying δ_ε to (1.57) and specializing to the particular case $\delta_\varepsilon q^i = \varepsilon Q^i(q,t)$, we obtain:

$$\delta_\varepsilon\, \tilde{q}^i = \delta_\varepsilon\Big(\exp(\delta_\varepsilon)\, q^i\Big) = \exp(\delta_\varepsilon)\Big(\delta_\varepsilon\, q^i\Big) = \varepsilon\, \exp(\delta_\varepsilon)\, Q^i(q,t) \overset{(1.59)}{=} \varepsilon\, Q^i(\tilde{q},t). \tag{1.62}$$

Owing to $\delta_\varepsilon f(q,t) = \varepsilon Q^j \frac{\partial}{\partial q^j} f(q,t)$, this gives the following differential equations for the $\tilde{q}^i(q,t)$:

$$Q^j(q,t)\, \frac{\partial \tilde{q}^i(q,t)}{\partial q^j} = Q^i(\tilde{q},t). \tag{1.63}$$

1.6 Exercises 5 – 9

Exercise 5: Derive the infinitesimal transformations $\delta_\varepsilon x$, $\delta_\varepsilon y$, $\delta_\varepsilon z$ arising from

$$a)\quad \tilde{t} = t, \quad \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \\ \tilde{z}(t) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$b)\quad \tilde{t} = t - \alpha, \quad \tilde{x}(\tilde{t}) = x(t), \quad \tilde{y}(\tilde{t}) = y(t), \quad \tilde{z}(\tilde{t}) = z(t)$$

Exercise 6: Consider the following infinitesimal transformations of $u(t)$ and $v(t)$:

$$\delta_\varepsilon u = \varepsilon\, v, \quad \delta_\varepsilon v = \varepsilon\, u \tag{1.64}$$

a) Compute $\exp(\delta_\varepsilon)u$ and $\exp(\delta_\varepsilon)v$ to third order in ε.

b) Guess or derive closed formulae for $\tilde{u} = \exp(\delta_\varepsilon)u$ and $\tilde{v} = \exp(\delta_\varepsilon)v$.

c) Write the differential equations (1.63) arising from (1.64).

d) Check whether your result of b) solves the equations obtained in c).

Exercise 7: Consider the following infinitesimal transformation of $u(t)$:

$$\delta_\varepsilon u = \varepsilon\, u^2 \tag{1.65}$$

a) Compute $\exp(\delta_\varepsilon)u$.

b) Write the differential equation (1.63) arising from (1.65).

c) Solve the equations obtained in b) and compare the solution to the result in a).

Exercise 8: This exercise serves to prove equation (1.59) for functions $f([q], t)$ that are polynomials or formal power series' in the q^i, $\dot{q}^i$, ...

a) Show that, for any g and h, $\exp(\delta_\varepsilon)$ applied to the product gh is equal to the product of $\exp(\delta_\varepsilon)g$ and $\exp(\delta_\varepsilon)h$:

$$\forall g, h: \quad \exp(\delta_\varepsilon)\,(g\,h) = \Big(\exp(\delta_\varepsilon)\,g\Big)\Big(\exp(\delta_\varepsilon)\,h\Big) \tag{1.66}$$

Hint: Use $\exp(\delta_\varepsilon) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(\delta_\varepsilon)^n$ on the left and right hand side of (1.66) and show that the coefficients of $((\delta_\varepsilon)^m g)((\delta_\varepsilon)^n h)$ agree on both sides for all m and n. In order to get an idea how it works you may first expand (1.66) to second order in ε.

b) Use (1.66) to show that (1.59) holds for all functions $f([q], t)$ that are polynomials or formal power series' in the q^i, $\dot{q}^i$, ... and may depend additionally explicitly on t.

Exercise 9: Show that $\delta_\varepsilon x = \varepsilon\, \dddot{x}$ is an infinitesimal symmetry of $L([x]) = \frac{m}{2}\,\dot{x}^2$.

Solutions of exercises 5 – 9

Exercise 5:

$$
\begin{aligned}
&a) \quad \delta_\varepsilon\, x = \varepsilon\, y, \quad \delta_\varepsilon\, y = -\varepsilon\, x, \quad \delta_\varepsilon\, z = 0 \\
&b) \quad \delta_\varepsilon\, x = \varepsilon\, \dot{x}, \quad \delta_\varepsilon\, y = \varepsilon\, \dot{y}, \quad \delta_\varepsilon\, z = \varepsilon\, \dot{z}
\end{aligned}
$$

Exercise 6:

$$
\begin{aligned}
a) \quad & \exp(\delta_\varepsilon)\, u = u + \varepsilon v + \tfrac{1}{2}\varepsilon^2 u + \tfrac{1}{6}\varepsilon^3 v + O(\varepsilon^4) = (1+\tfrac{1}{2}\varepsilon^2)\, u + (\varepsilon + \tfrac{1}{6}\varepsilon^3)\, v + O(\varepsilon^4) \\
& \exp(\delta_\varepsilon)\, v = v + \varepsilon u + \tfrac{1}{2}\varepsilon^2 v + \tfrac{1}{6}\varepsilon^3 u + O(\varepsilon^4) = (1+\tfrac{1}{2}\varepsilon^2)\, v + (\varepsilon + \tfrac{1}{6}\varepsilon^3)\, u + O(\varepsilon^4) \\
b) \quad & \tilde{u} = u\, \cosh\varepsilon + v\, \sinh\varepsilon, \quad \tilde{v} = v\, \cosh\varepsilon + u\, \sinh\varepsilon \\
c) \quad & v\, \frac{\partial\tilde{u}}{\partial u} + u\, \frac{\partial\tilde{u}}{\partial v} = \tilde{v}, \quad v\, \frac{\partial\tilde{v}}{\partial u} + u\, \frac{\partial\tilde{v}}{\partial v} = \tilde{u} \\
d) \quad & v\, \frac{\partial\tilde{u}}{\partial u} + u\, \frac{\partial\tilde{u}}{\partial v} = v\, \cosh\varepsilon + u\, \sinh\varepsilon = \tilde{v}, \quad v\, \frac{\partial\tilde{v}}{\partial u} + u\, \frac{\partial\tilde{v}}{\partial v} = v\, \sinh\varepsilon + u\, \cosh\varepsilon = \tilde{u}, \quad \text{o.k.}
\end{aligned}
$$

Exercise 7:

$$
\begin{aligned}
a) \quad & \tilde{u} = u + \varepsilon u^2 + \varepsilon^2 u^3 + \varepsilon^3 u^4 + \cdots = \sum_{n=0}^{\infty} u\, (\varepsilon u)^n = \frac{u}{1-\varepsilon u} \\
b) \quad & u^2\, \frac{d\tilde{u}}{du} = \tilde{u}^2 \quad \left(\text{here } \frac{\partial}{\partial u} \text{ was replaced by } \frac{d}{du} \text{ because of } \tilde{u} = \tilde{u}(u)\right) \\
c) \quad & u^2\, \frac{d\tilde{u}}{du} = \tilde{u}^2 \;\Rightarrow\; \frac{d\tilde{u}}{\tilde{u}^2} = \frac{du}{u^2} \;\Rightarrow\; \frac{1}{\tilde{u}} = \frac{1}{u} + C,\; C = \mathit{constant} \\
& \Rightarrow\; \tilde{u} = \frac{1}{\frac{1}{u} + C} = \frac{u}{1 + Cu}\;. \quad \text{This agrees with a) for } C = -\varepsilon.
\end{aligned}
$$

Exercise 8:
a) ε^n-terms in $\exp(\delta_\varepsilon)(gh)$:

$$
\frac{1}{n!}\,(\delta_\varepsilon)^n\,(g\,h) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\Big((\delta_\varepsilon)^k\,g\Big)\Big((\delta_\varepsilon)^{n-k}\,h\Big) \tag{1.67}
$$

ε^n-terms in $\Big(\exp(\delta_\varepsilon)\,g\Big)\Big(\exp(\delta_\varepsilon)\,h\Big)$:

$$
\sum_{k=0}^{n}\Big(\frac{1}{k!}(\delta_\varepsilon)^k\,g\Big)\Big(\frac{1}{(n-k)!}(\delta_\varepsilon)^{n-k}\,h\Big) \tag{1.68}
$$

Owing to $\frac{1}{n!}\binom{n}{k} = \frac{1}{k!\,(n-k)!}$, (1.67) is equal to (1.68) which proves (1.66).

b) (1.66) implies that (1.59) holds for arbitrary monomials in the q^i, $\dot{q}^i$, …(e.g., for $g = q^1$ and $h = q^2$, (1.66) yields $\exp(\delta_\varepsilon)(q^1q^2) = (\exp(\delta_\varepsilon)q^1)(\exp(\delta_\varepsilon)q^2)$; for $g = q^1q^2$ and $h = \dot{q}^1$, it yields $\exp(\delta_\varepsilon)(q^1q^2\dot{q}^1) = (\exp(\delta_\varepsilon)(q^1q^2))(\exp(\delta_\varepsilon)\dot{q}^1) = (\exp(\delta_\varepsilon)q^1)(\exp(\delta_\varepsilon)q^2)(\exp(\delta_\varepsilon)\dot{q}^1)$ etc). Owing to $\exp(\delta_\varepsilon)(a+b) = \exp(\delta_\varepsilon)a + \exp(\delta_\varepsilon)b$ (for all a, b) (1.59) holds also for arbitrary linear combinations of such monomials and thus for polynomials and formal power series' in the q^i, $\dot{q}^i$, … with coefficients which can depend arbitrarily on t (recall that $\delta_\varepsilon t = 0$).

Exercise 9:

$$
\delta_\varepsilon L([x]) = m\,\varepsilon\;\dot{x}\;\ddddot{x} = m\,\varepsilon\,\frac{d}{dt}(\dot{x}\;\dddot{x} - \tfrac{1}{2}\,\ddot{x}^2)
$$

1.7 Constants of motion

A constant of motion of a mechanical system is a function $J([q],t)$ that is constant *for every solution of the equations of motion.* This motivates the following definition which is not restricted to Lagrangean systems:

Definition: Let $\mathcal{L}_i([q],t) = 0$ denote the equations of motion of a mechanical system (for Lagrangean systems one has $\mathcal{L}_i = \frac{\delta L}{\delta q^i}$). A function J of the q^i, $\dot{q}^i$, ..., t is called a constant of motion (or conserved quantity) of the system if its total time derivative is a linear combination of the $\mathcal{L}_i$, $\dot{\mathcal{L}}_i$, ... with coefficient functions that may depend on the q^i, $\dot{q}^i$, ..., t:

$$\frac{dJ([q],t)}{dt} = G^i \mathcal{L}_i([q],t), \quad G^i = g^{i(0)}([q],t) + g^{i(1)}([q],t)\frac{d}{dt} + g^{i(2)}([q],t)\frac{d^2}{dt^2} + \dots \tag{1.69}$$

Example: In this example we use standard 3D vector notation (scalar product: $\vec{a}\cdot\vec{b}$, vector product: $\vec{a}\times\vec{b}$). We treat the motion of a point mass m in a Coulomb potential with Lagrangian

$$L([\vec{r}]) = \frac{m}{2}\dot{\vec{r}}\cdot\dot{\vec{r}} + \frac{k}{r}, \quad r = \sqrt{\vec{r}\cdot\vec{r}}, \quad \vec{r} = (x,y,z). \tag{1.70}$$

The equations of motion deriving from this Lagrangian read $\mathcal{L}_x = 0$, $\mathcal{L}_y = 0$, $\mathcal{L}_z = 0$ with

$$\mathcal{L}_x = \frac{\delta L}{\delta x} = -\frac{kx}{r^3} - m\ddot{x}, \quad \mathcal{L}_y = \frac{\delta L}{\delta y} = -\frac{ky}{r^3} - m\ddot{y}, \quad \mathcal{L}_z = \frac{\delta L}{\delta z} = -\frac{kz}{r^3} - m\ddot{z}.$$

Let $\vec{\mathcal{L}}$ denote the vector with components $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$:

$$\vec{\mathcal{L}} = (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z) = -\frac{k\vec{r}}{r^3} - m\ddot{\vec{r}}.$$

Well-known constants of motion are the energy E, the components of the angular momentum $\vec{L}$ and the components of the Runge-Lenz vector $\vec{R}$, with

$$E = \frac{m}{2}\dot{\vec{r}}\cdot\dot{\vec{r}} - \frac{k}{r} \tag{1.71}$$

$$\vec{L} = m\,\vec{r}\times\dot{\vec{r}} \tag{1.72}$$

$$\vec{R} = \dot{\vec{r}}\times\vec{L} - \frac{k\vec{r}}{r} \tag{1.73}$$

Each of these quantities verifies equation (1.69) because its total time derivative is linear in the components of $\vec{\mathcal{L}}$:[3]

$$\dot{E} = m\,\ddot{\vec{r}}\cdot\dot{\vec{r}} + \frac{k\vec{r}\cdot\dot{\vec{r}}}{r^3} = -\vec{\mathcal{L}}\cdot\dot{\vec{r}} \tag{1.74}$$

$$\dot{\vec{L}} = -\vec{r}\times\vec{\mathcal{L}} \tag{1.75}$$

$$\dot{\vec{R}} = -2\vec{r}(\vec{\mathcal{L}}\cdot\dot{\vec{r}}) + \dot{\vec{r}}(\vec{\mathcal{L}}\cdot\vec{r}) + \vec{\mathcal{L}}(\dot{\vec{r}}\cdot\vec{r}) = \vec{\mathcal{L}}\times(\dot{\vec{r}}\times\vec{r}) + \dot{\vec{r}}\times(\vec{\mathcal{L}}\times\vec{r}) \tag{1.76}$$

1.8 Noethers first theorem – simple version

Noethers first theorem establishes, for Lagrangean mechanical systems, a correspondence of infinitesimal global symmetries and constants of motion. We shall now derive the simple ('standard') version of the theorem, which however does not yet provide a *bijective* correspondence of symmetries and constants of motion (see comments below).

[3] The computation of $\dot{\vec{L}}$ and $\dot{\vec{R}}$ is given in section 1.9.

a) We assume that δ_ε is a symmetry of $S[q] = \int dt\, L([q],t)$ and shall show that δ_ε gives rise to a constant of motion. For definiteness and simplicity of formulae we shall assume $L = L(q^i, \dot{q}^i, t)$ (it is straightforward to generalize the reasoning to the case that L depends on higher order time derivatives of the q^i as well).

Our assumption that δ_ε is a symmetry means

$$\delta_\varepsilon L = \frac{dK}{dt} \tag{1.77}$$

(for some K). For every $\delta_\varepsilon L(q^i, \dot{q}^i, t)$ one has:

$$\begin{aligned} \delta_\varepsilon L &= (\delta_\varepsilon q^i)\, \frac{\partial L}{\partial q^i} + (\delta_\varepsilon \dot{q}^i)\, \frac{\partial L}{\partial \dot{q}^i} \\ &= (\delta_\varepsilon q^i)\, \frac{\partial L}{\partial q^i} + \frac{d}{dt}\Big((\delta_\varepsilon q^i)\, \frac{\partial L}{\partial \dot{q}^i}\Big) - (\delta_\varepsilon q^i)\, \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \\ &= (\delta_\varepsilon q^i)\, \frac{\delta L}{\delta q^i} + \frac{d}{dt}\Big((\delta_\varepsilon q^i)\, \frac{\partial L}{\partial \dot{q}^i}\Big). \end{aligned} \tag{1.78}$$

Combining (1.77) and (1.78) gives

$$\frac{dJ}{dt} = (\delta_\varepsilon q^i)\, \frac{\delta L}{\delta q^i}, \quad J = K - (\delta_\varepsilon q^i)\, \frac{\partial L}{\partial \dot{q}^i}\ . \tag{1.79}$$

Hence, J given by $K - (\delta_\varepsilon q^i)\, \frac{\partial L}{\partial \dot{q}^i}$ is a constant of motion.

b) Now we assume that J is a constant of motion of a system described by an action $S[q] = \int dt\, L([q],t)$. This means that

$$\dot{J} = (g^{i(0)} + g^{i(1)} \frac{d}{dt} + \dots)\, \frac{\delta L}{\delta q^i}\ . \tag{1.80}$$

We shall show that this corresponds to an infinitesimal symmetry of $S[q]$. As a first step we rewrite the r.h.s. of (1.80):

$$\dot{J} = (g^{i(0)} - \dot{g}^{i(1)} + \dots)\, \frac{\delta L}{\delta q^i} + \frac{d}{dt}\Big(g^{i(1)}\, \frac{\delta L}{\delta q^i} + \dots\Big). \tag{1.81}$$

In the next step we bring the total time derivative on the r.h.s. of (1.81) to the l.h.s.:

$$\dot{J}' = Q^i\, \frac{\delta L}{\delta q^i}\ , \quad J' = J - g^{i(1)}\, \frac{\delta L}{\delta q^i} + \dots, \quad Q^i = g^{i(0)} - \dot{g}^{i(1)} + \dots \tag{1.82}$$

As a third step we rewrite $Q^i\, \frac{\delta L}{\delta q^i}$:

$$\begin{aligned} Q^i\, \frac{\delta L}{\delta q^i} &= Q^i \Big(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\Big) \\ &= Q^i\, \frac{\partial L}{\partial q^i} - \frac{d}{dt}\Big(Q^i\, \frac{\partial L}{\partial \dot{q}^i}\Big) + \dot{Q}^i\, \frac{\partial L}{\partial \dot{q}^i}\ . \end{aligned} \tag{1.83}$$

Finally we use (1.83) in (1.82), add to the resultant equation the total time derivative $\frac{d}{dt}\Big(Q^i\, \frac{\partial L}{\partial \dot{q}^i}\Big)$ (in order to move that term to the l.h.s. of the equation) and multiply it by ε. This gives

$$\begin{aligned} &\dot{K} = \varepsilon\, Q^i\, \frac{\partial L}{\partial q^i} + \varepsilon\, \dot{Q}^i\, \frac{\partial L}{\partial \dot{q}^i} = \delta_\varepsilon L, \\ &K = \varepsilon\Big(J' + Q^i\, \frac{\partial L}{\partial \dot{q}^i}\Big), \quad \delta_\varepsilon q^i = \varepsilon Q^i = \varepsilon\, (g^{i(0)} - \dot{g}^{i(1)} + \dots). \end{aligned} \tag{1.84}$$

Hence, δ_ε is an infinitesimal symmetry of $S[q]$.

Comments

- Notice that in the above derivation of Noethers first theorem we only used (repeatedly) $f\dot{g} = \frac{d}{dt}(fg) - \dot{f}g$. Hence, this version of Noethers first theorem appears to be a banality. However, it is important to realize that it is a banality only because we used the definition of symmetries according to (1.16) and, in particular, the definition of constants of motion according to (1.69). Hence, this version of Noethers first theorem actually rests on the insight that symmetries and, especially, constants of motion can be formulated according to (1.16) and (1.69), respectively.
- The above version of Noethers first theorem does not establish a bijective correspondence of symmetries and constants of motion, for:
  - It can happen that $J = K - (\delta_\varepsilon q^i)\,\frac{\partial L}{\partial \dot{q}^i}$ (see (1.79)) vanishes identically. Hence, there are symmetries which do not correspond to constants of motion. Example:

$$L = \frac{m}{2}\,(\dot{x}^2 + \dot{y}^2), \quad \delta_\varepsilon x = \varepsilon \ddot{y}, \quad \delta_\varepsilon y = -\varepsilon \ddot{x}$$
$$\Rightarrow \quad \delta_\varepsilon L = \varepsilon\, m\, \frac{d}{dt}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) \quad \Rightarrow \quad J = 0. \tag{1.85}$$

  - Constant functions, such as $J = 1$, do not correspond to symmetries. Hence, there are constants of motion which do not correspond to symmetries (notice that constant functions are constants of motion according to our definition).
- A bijective correspondence between symmetries and constants of motion is obtained for equivalence classes of symmetries and constants of motion where two symmetries are called equivalent if they differ by a trivial symmetry, and two constants of motion are called equivalent if they differ by a trivial constant of motion. A trivial symmetry transformation vanishes for all solutions of the equations of motion up to a gauge transformation, a trivial constant of motion is a linear combination of the $\mathcal{L}_i$ and their total time derivatives with coefficient functions that may depend on the $t, q^i, \dot{q}^i, \ldots$ plus a constant function (i.e., a trivial constant of motion equals a constant function for all solutions of the equations of motion). This will be explained later in more detail.

1.9 Exercise 10 - Motion in the Coulomb potential: Conservation of angular momentum and Runge-Lenz vector, and the corresponding symmetries

We treat again the system with Lagrangian (1.70), i.e., a point mass m in a Coulomb potential $U(r) = -k/r$ and shall use the same notation as in the example in section 1.7.

a) Verify that the components of $\vec{L}$ and $\vec{R}$ given in equations (1.72) and (1.73) respectively are constants of motion by deriving equations (1.75) and (1.76).

b) Determine the infinitesimal symmetry transformations which correspond to the conservation of the third components L_z and R_z of $\vec{L}$ and $\vec{R}$, respectively.

c) How is the result of b) contained in the following infinitesimal transformations $\delta_{\vec{\varepsilon}}$? (Guess the notation used here!)

$$\delta_{\vec{\varepsilon}}\vec{r} = \vec{\varepsilon} \times (\vec{r} \times \dot{\vec{r}}) + \vec{r} \times (\vec{\varepsilon} \times \dot{\vec{r}})$$

d) Verify explicitly that the transformations $\delta_{\vec{\varepsilon}}$ generate symmetries of the action, and use the result to conclude that these symmetries correspond via Noethers first theorem to the constants of motion given by the components of $\vec{R}$.

Solution

a) Direct computations give:

$$
\begin{aligned}
\dot{\vec{L}} &= m\,\vec{r}\times\ddot{\vec{r}} = -\vec{r}\times\vec{\mathcal{L}}\\
\dot{\vec{R}} &= \ddot{\vec{r}}\times\vec{L} + \dot{\vec{r}}\times\dot{\vec{L}} - \frac{k\dot{\vec{r}}}{r} + k\vec{r}\,\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}\\
&= -\frac{1}{m}\Big(\vec{\mathcal{L}} + \frac{k\vec{r}}{r^3}\Big)\times\vec{L} - \dot{\vec{r}}\times(\vec{r}\times\vec{\mathcal{L}}) - \frac{k\dot{\vec{r}}}{r} + k\vec{r}\,\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}\\
&= -\Big(\vec{\mathcal{L}} + \frac{k\vec{r}}{r^3}\Big)\times(\vec{r}\times\dot{\vec{r}}) - \dot{\vec{r}}\times(\vec{r}\times\vec{\mathcal{L}}) - \frac{k\dot{\vec{r}}}{r} + k\vec{r}\,\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}\\
&= -\vec{r}\Big(\vec{\mathcal{L}}\cdot\dot{\vec{r}} + \frac{k\vec{r}\cdot\dot{\vec{r}}}{r^3}\Big) + \dot{\vec{r}}\Big(\vec{\mathcal{L}}\cdot\vec{r} + \frac{k\vec{r}\cdot\vec{r}}{r^3}\Big) - \vec{r}(\dot{\vec{r}}\cdot\vec{\mathcal{L}}) + \vec{\mathcal{L}}(\dot{\vec{r}}\cdot\vec{r}) - \frac{k\dot{\vec{r}}}{r} + k\vec{r}\,\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}\\
&= -2\vec{r}\,(\vec{\mathcal{L}}\cdot\dot{\vec{r}}) + \dot{\vec{r}}\,(\vec{\mathcal{L}}\cdot\vec{r}) + \vec{\mathcal{L}}\,(\dot{\vec{r}}\cdot\vec{r})
\end{aligned}
$$

b) The expressions for $\dot{L}_z$ and $\dot{R}_z$ contained in equations (1.75) and (1.76) are of the form $\dot{J} = Q^i\mathcal{L}_i$ where Q^i are *functions* rather than operators (i.e., in this case (1.69) reduces to $G^i = g^{i(0)} \equiv Q^i$). Whenever this is the case, the infinitesimal transformations corresponding to J are simply given by $\delta_\varepsilon q^i = \varepsilon Q^i$, cf. section 1.8 (especially equation (1.84)). Hence, in the present case one can read off the transformations of x, y, z from the coefficients of $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$ in $\dot{L}_z$ and $\dot{R}_z$, respectively.

$$
\begin{aligned}
L_z : \dot{L}_z &= -x\mathcal{L}_y + y\mathcal{L}_x \quad\Rightarrow\quad \delta_\varepsilon x = \varepsilon y,\quad \delta_\varepsilon y = -\varepsilon x,\quad \delta_\varepsilon z = 0\\
R_z : \dot{R}_z &= -2z(\vec{\mathcal{L}}\cdot\dot{\vec{r}}) + \dot{z}(\vec{\mathcal{L}}\cdot\vec{r}) + \mathcal{L}_z(\dot{\vec{r}}\cdot\vec{r})\\
&= \mathcal{L}_x(-2z\dot{x} + \dot{z}x) + \mathcal{L}_y(-2z\dot{y} + \dot{z}y) + \mathcal{L}_z(-2z\dot{z} + \dot{z}z + \dot{x}x + \dot{y}y + \dot{z}z)\\
&= \mathcal{L}_x(-2z\dot{x} + \dot{z}x) + \mathcal{L}_y(-2z\dot{y} + \dot{z}y) + \mathcal{L}_z(\dot{x}x + \dot{y}y) \quad\Rightarrow\\
\delta_\varepsilon x &= \varepsilon(-2z\dot{x} + \dot{z}x)\\
\delta_\varepsilon y &= \varepsilon(-2z\dot{y} + \dot{z}y)\\
\delta_\varepsilon z &= \varepsilon(\dot{x}x + \dot{y}y)
\end{aligned}
$$

c) $\vec{\varepsilon}$ corresponds to $\vec{R}$, i.e., ε_z corresponds to R_z. Setting $\vec{\varepsilon} = (0, 0, \varepsilon)$, one obtains

$$
\begin{aligned}
\delta_{(0,0,\varepsilon)}\vec{r} &= \vec{\varepsilon}\times(\vec{r}\times\dot{\vec{r}}) + \vec{r}\times(\vec{\varepsilon}\times\dot{\vec{r}})\\
&= \vec{r}(\vec{\varepsilon}\cdot\dot{\vec{r}}) + \vec{\varepsilon}(\vec{r}\cdot\dot{\vec{r}}) - 2\dot{\vec{r}}(\vec{\varepsilon}\cdot\vec{r})\\
&= \vec{r}\,\varepsilon\dot{z} + (0, 0, \varepsilon)(\dot{x}x + \dot{y}y + \dot{z}z) - 2\dot{\vec{r}}\,\varepsilon z\\
&= \varepsilon\,(x\dot{z} - 2\dot{x}z, y\dot{z} - 2\dot{y}z, \dot{x}x + \dot{y}y)
\end{aligned}
$$

d) A somewhat lengthy calculation yields:

$$
\delta_{\vec{\varepsilon}} L = \dot{K},\quad K = \vec{\varepsilon}\cdot\Big(\vec{L}\times\dot{\vec{r}} - \frac{k\vec{r}}{r}\Big).
$$

Owing to $L = L(q^i, \dot{q}^i)$, the corresponding constants of motion are $J = K - \frac{\partial L}{\partial \dot{q}^i}\,\delta_\varepsilon q^i$, cf. (1.79). This gives

$$
\begin{aligned}
J &= K - \frac{\partial L}{\partial \dot{x}}\,\delta_{\vec{\varepsilon}}\,x - \frac{\partial L}{\partial \dot{y}}\,\delta_{\vec{\varepsilon}}\,y - \frac{\partial L}{\partial \dot{z}}\,\delta_{\vec{\varepsilon}}\,z\\
&= K - 2\vec{\varepsilon}\cdot(\vec{L}\times\dot{\vec{r}}) = \vec{\varepsilon}\cdot\Big(\dot{\vec{r}}\times\vec{L} - \frac{k\vec{r}}{r}\Big) = \vec{\varepsilon}\cdot\vec{R}
\end{aligned}
$$