Seiberg-Witten maps and anomalies in noncommutative Yang-Mills theories

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- Existence, construction and ambiguities of SW maps
- Gauge anomalies and anomaly freedom in NC YM theories
- BRST cohomological perspective on both topics

Refs.: G. Barnich, FB, M. Grigoriev, hep-th/0308092
FB, C.P. Martin, F. Ruiz Ruiz, hep-th/0307292

Introduction

Weyl-Moyal product:

$$f_1 \star f_2 = f_1 \, \exp(\overleftarrow{\partial_\mu} \, \tfrac{\mathrm{i}}{2} \, g \, \theta^{\mu\nu} \, \overrightarrow{\partial}_\nu) \, f_2, \qquad \theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constant}$$

Action:

$$\hat{I}[\hat{A}] = -\tfrac{1}{4} \int d^n x \, \mathrm{Tr}\,(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}), \qquad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + [\hat{A}_\mu \overset{\star}{,} \hat{A}_\nu]$$

Gauge transformations:

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} + [\hat{A}_\mu \overset{\star}{,} \hat{\lambda}] \equiv \hat{D}_\mu \hat{\lambda}$$

SW maps:

$$\hat{A}_\mu = \hat{A}_\mu(A, g) = A_\mu + O(g), \qquad \hat{\lambda} = \hat{\lambda}(\lambda, A, g) = \lambda + O(g)$$

with A_μ and λ Lie algebra valued, such that

$$\delta_\lambda A_\mu = \partial_\mu \lambda + [A_\mu, \lambda] \equiv D_\mu \lambda \;\; \Rightarrow \;\; \boxed{\delta_\lambda \hat{A}_\mu(A, g) = (\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu)(A, \lambda, g)}$$

Consistent deformations

Consider an action $I^{(0)}[\varphi]$ with gauge invariance $\delta_\lambda^{(0)}$, i.e. $\delta_\lambda^{(0)} I^{(0)}[\varphi] = 0$.

Consistent deformations of $I^{(0)}[\varphi]$, $\delta_\lambda^{(0)}$:

$$I[\varphi, g] = I^{(0)}[\varphi] + \sum_{k\geq 1} g^k I^{(k)}[\varphi], \quad \delta_\lambda = \delta_\lambda^{(0)} + \sum_{k\geq 1} g^k \delta_\lambda^{(k)}, \quad \delta_\lambda I[\varphi, g] = 0$$

Equivalent deformations: related by field redefinitions $\widehat{\varphi}(\varphi, g)$, $\widehat{\lambda}(\lambda, \varphi, g)$:

$$\widehat{I}[\widehat{\varphi}(\varphi, g), g] = I[\varphi, g], \quad (\widehat{\delta}_{\widehat{\lambda}} \widehat{\varphi})(\varphi, \lambda, g) \approx \delta_\lambda \widehat{\varphi}(\varphi, g)$$

Trivial deformations: $I \sim I^{(0)}$ and $\delta \sim \delta^{(0)}$

Two types of nontrivial deformations:

Type I: $I \not\sim I^{(0)}$, $\delta \sim \delta^{(0)}$

Type II: $I \not\sim I^{(0)}$, $\delta \not\sim \delta^{(0)}$

SW map: NC YM theories are type I deformations of YM theories

Result on deformations of YM theories [G. Barnich, FB, M. Henneaux 1994, 2000]

Pure YM theories (including eff. theories):

- Semisimple gauge group:
  only consistent deformations of type I, i.e.:
  no nontrivial deformations of gauge transformations at all!

- Gauge group with abelian factors, esp. free theories (1st order result):

$$I^{(1)} \sim I^{(1)}_{\text{inv}} + \int d^n x \, (\underbrace{k_i^{\Delta} A^i_\mu j^\mu_\Delta}_{\text{Noether type}} - \underbrace{\tfrac{1}{2} f_{ijk} F^{\mu\nu i} A^j_\mu A^k_\nu}_{\text{YM type}})$$

$$+ \int d^n x \, \underbrace{k_{ijk}[-\tfrac{1}{2} F^{\mu\nu i} A^j_\mu A^k_\nu + \tfrac{2}{n-4} x^\mu (F^i_{\mu\rho} F^{\nu\rho j} - \tfrac{1}{4} \delta^\nu_\mu F^i_{\rho\sigma} F^{\rho\sigma j}) A^k_\nu]}_{\text{peculiar type } (n \neq 4)}$$

where

$I^{(1)}_{\text{inv}}$: $\int$ G-inv. polynomials in $F, DF, \ldots$ plus Chern-Simons terms in odd dims.

A^i_μ : abelian gauge fields

j^μ_Δ : gauge invariant conserved currents of the undeformed theory

$k_i^\Delta, f_{ijk}, k_{ijk}$: constants with $f_{ijk} = f_{[ijk]}$, $k_{ijk} = -k_{ikj} = k_{jik}$

Corresponding first order deformations of the gauge transformations:

$$\delta_\lambda^{(1)} A_\mu^a \sim k_i^{\Delta}\,\lambda^i G^a_{\Delta\mu} \quad \text{if } A_\mu^a \text{ nonabelian}$$

$$\delta_\lambda^{(1)} A_\mu^i \sim \underbrace{k_j^{\Delta}\,\lambda^j G^i_{\Delta\mu}}_{\text{Noether type}} + \underbrace{f_{ijk} A_\mu^j \lambda^k}_{\text{YM type}} + \underbrace{k_{ijk}(A_\mu^j + \tfrac{2}{n-4}\, x^\nu F^j_{\nu\mu})\lambda^k}_{\text{peculiar type } (n\neq 4)}$$

where $G^a_{\Delta\mu}$ is the (infinitesimal) transformation of A_μ^a under the global symmetry of $I^{(0)}$ that corresponds to j^μ_Δ:

$$G^a_{\Delta\mu}\,\frac{\delta I^{(0)}}{\delta A_\mu^a} = \partial_\mu j^\mu_\Delta$$

- Deformed gauge transformations do not involve derivatives of λ^i $\Rightarrow$ existence of SW map for $U(N)$ NC YM theories

BRST-cohomological approach to consistent deformations and SW maps

Field-antifield formalism:

Fields: $\{\phi\} = \{\varphi, C\} \overset{\text{paired}}{\longleftrightarrow}$ antifields: $\{\phi^*\} = \{\varphi^*, C^*\}$

Antibracket: $(F, G) = \int d^n x\, F\Big(\frac{\overleftarrow{\delta}}{\delta\phi}\,\frac{\overrightarrow{\delta}}{\delta\phi^*} - \frac{\overleftarrow{\delta}}{\delta\phi^*}\,\frac{\overrightarrow{\delta}}{\delta\phi}\Big)G$

Master action: $S[\phi, \phi^*] = \underbrace{I[\varphi]}_{\text{ordinary action}} + \underbrace{\int d^n x\, \varphi^* \delta_C \varphi}_{\text{gauge transformations}} + \underbrace{\ldots}_{\text{gauge algebra}}$ such that $\underbrace{(S, S) = 0}_{\text{master equation}}$

BRST differential (coboundary operator): $s = (S,\ \cdot\ )$ $\quad (\Rightarrow\ s^2 = 0)$

Consistent deformations: $S = S^{(0)} + \sum_{k\geq 1} g^k S^{(k)}$

Consistent deformations and local BRST-cohomology:

Master equation: $(S,S)=0 \quad \overset{\frac{\partial}{\partial g}}{\Rightarrow} \quad \left(S,\frac{\partial S}{\partial g}\right)=0 \quad \Leftrightarrow \quad \boxed{s\,\frac{\partial S}{\partial g}=0}$

Field redefinitions $\longrightarrow$ anticanonical transformations $\widehat{\phi}(\phi,\phi^*,g)$, $\widehat{\phi}^*(\phi,\phi^*,g)$:

$$\frac{d\widehat{\phi}}{dg}=(\Xi,\widehat{\phi}),\quad \frac{d\widehat{\phi}^*}{dg}=(\Xi,\widehat{\phi}^*)\quad \Rightarrow \quad \frac{d}{dg}S(\widehat{\phi},\widehat{\phi}^*,g)=\frac{\partial S}{\partial g}-(S,\Xi)=\frac{\partial S}{\partial g}-s\,\Xi$$

Equivalent deformations: $\boxed{S\sim S' \quad \Rightarrow \quad \frac{\partial S}{\partial g}-\frac{\partial S'}{\partial g}=s\,\Xi}$

Hence: consistent deformations are determined by the cohomology $H(s)$

BRST-cohomological description of SW maps:

$$\{\widehat{\phi}\} = \{\widehat{A}_\mu, \widehat{C}\}, \quad \{\phi\} = \{A_\mu, C\}$$

Master action for NC gauge theories:

$$S[\widehat{\phi}, \widehat{\phi}^*, g] = \int d^n x \Big[-\tfrac{1}{4}\,\mathrm{Tr}\,(\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}) + \widehat{A}^{*\mu} \star \widehat{D}_\mu \widehat{C} + \widehat{C}^* \star (\widehat{C} \star \widehat{C}) \Big]$$

SW map:

$$S[\widehat{\phi}(\phi,\phi^*,g), \widehat{\phi}^*(\phi,\phi^*,g), g] = \underbrace{I_{\mathrm{eff}}[A,g]}_{\text{no antifields}} + \int d^n x \Big[\underbrace{A^{*\mu} D_\mu C + C^* C C}_{\text{no dependence on } g} \Big]$$

$$\overset{\frac{d}{dg}}{\Longrightarrow} \quad \frac{\partial S}{\partial g} - s\,\Xi = \frac{\partial I_{\mathrm{eff}}[A,g]}{\partial g} \quad \text{(antifield dependence of } \frac{\partial S}{\partial g} \text{ is trivial)}$$

$$\frac{d\widehat{\phi}}{dg} = (\Xi, \widehat{\phi}), \qquad \frac{d\widehat{\phi}^*}{dg} = (\Xi, \widehat{\phi}^*) \quad \text{(diff. eqs. for SW map)}$$

Remark: for gauge group with Lie algebra $\mathfrak{g} \neq \mathfrak{u}(N)$, this applies to fields valued in enveloping algebra of $\mathfrak{g}$. Fields and antifields A, C, A^*, C^* that do not belong to $\mathfrak{g}$ are set to zero in the end.

Hence: SW map $\Leftrightarrow$ $$\frac{\partial S}{\partial g}=s\,\Xi+\underbrace{\text{terms without antifields}}_{dI_{\text{eff}}/dg}$$

Existence, explicit construction and ambiguities of SW maps:

$$\frac{\partial S}{\partial g}=\frac{\mathrm{i}\theta^{\alpha\beta}}{2}\int d^nx\,\mathrm{Tr}\,(-\widehat{F}^{\mu\nu}\star\partial_\alpha\widehat{A}_\mu\star\partial_\beta\widehat{A}_\nu+\widehat{A}^{*\mu}\star\{\partial_\alpha\widehat{A}_\mu\stackrel{\star}{,}\partial_\beta\widehat{C}\}+\widehat{C}^*\star\partial_\alpha\widehat{C}\star\partial_\beta\widehat{C})$$

Existence: $\Leftarrow$ terms with antifields depend on $\widehat{C}$ only via derivatives

Explicit construction: determination of Ξ (contracting homotopy)

Result: $\Xi=\frac{\mathrm{i}}{4}\theta^{\alpha\beta}\int d^nx\,\mathrm{Tr}\,(-\widehat{A}^{*\mu}\{\widehat{F}_{\alpha\mu}+\partial_\alpha\widehat{A}_\mu\stackrel{\star}{,}\widehat{A}_\beta\}+\widehat{C}^*\{\widehat{A}_\alpha\stackrel{\star}{,}\partial_\beta\widehat{C}\})$

$$\frac{d\widehat{A}_\mu}{dg}=(\Xi,\widehat{A}_\mu)=\frac{\mathrm{i}}{4}\theta^{\alpha\beta}\{\widehat{F}_{\alpha\mu}+\partial_\alpha\widehat{A}_\mu\stackrel{\star}{,}\widehat{A}_\beta\},\qquad\frac{\partial\widehat{C}}{\partial g}=(\Xi,\widehat{C})=-\frac{\mathrm{i}}{4}\theta^{\alpha\beta}\{\widehat{A}_\alpha\stackrel{\star}{,}\partial_\beta\widehat{C}\}$$

$$\frac{dI_{\text{eff}}[\widehat{A}(A,g),g]}{dg}=\mathrm{i}\theta^{\alpha\beta}\int d^nx\,\mathrm{Tr}\,(\tfrac{1}{8}\widehat{F}_{\alpha\beta}\star\widehat{F}_{\mu\nu}\star\widehat{F}^{\mu\nu}-\tfrac{1}{2}\widehat{F}_{\alpha\mu}\star\widehat{F}_{\beta\nu}\star\widehat{F}^{\mu\nu})$$

Ambiguities: determined by $0=s\,(\Delta\Xi)+\underbrace{\text{terms without antifields}}_{d(\Delta I_{\text{eff}})/dg}$

Resultant ambiguities of SW map:

$$\widehat{A}_\mu(A,g) = \left[\Lambda^{-1}\star\widehat{A}_\mu^{sp}\star\Lambda+\Lambda^{-1}\star\partial_\mu\Lambda\right]_{A_\mu\to A'_\mu(A,g)}$$
$$\widehat{\lambda}(\lambda,A,g) = \left[\Lambda^{-1}\star\widehat{\lambda}^{sp}\star\Lambda+\Lambda^{-1}\star\delta_\lambda\Lambda\right]_{A_\mu\to A'_\mu(A,g)}$$

where

$\Lambda(A,g)=\exp_\star(f^B(A,g)T_B)$ with arbitrary $f^B(A,g)$

$\widehat{A}_\mu^{sp}(A,g)$, $\widehat{\lambda}^{sp}(\lambda,A,g)$ particular SW map

${A'}_\mu^B(A,g)=[A_\mu+W_\mu(A,g)]^C R_C^B(g)$

$\delta_\lambda W_\mu(A,g)=[W_\mu(A,g),\lambda]$ (i.e., gauge covariant)

$T_B\to R_B^C(g)T_C$ (outer) Lie algebra automorphism

Hence, SW map is determined only up to (compositions of)

- noncommutative gauge transformations of $\widehat{A}_\mu$
- gauge covariant shifts of enveloping algebra valued A_μ
- (outer) automorphisms of Lie(enveloping algebra)

Gauge anomalies

Chiral gauge anomalies in NC gauge theories ($n = 4$, 1-loop):

$$\mathcal{A}[\widehat{C}, \widehat{A}, g] = \int \mathrm{Tr}[\widehat{C} \star d(\widehat{A} \star d\widehat{A} + \tfrac{1}{2}\widehat{A} \star \widehat{A} \star \widehat{A})]$$

Puzzle: vanishing of $\mathcal{A}$ seems to impose

$$\mathrm{Tr}(T_a T_b T_c) = 0 \quad \text{rather than} \quad \mathrm{Tr}(T_{(a} T_b T_{c)}) = 0$$

On the other hand:

SW map $\Rightarrow$ candidate anomalies are known: same as in (effective) YM theories [G. Barnich, FB, M. Henneaux '94]:

$$\text{I.} \quad \int \mathrm{Tr}[C d(A dA + \tfrac{1}{2}A^3)]$$

$$\text{II.} \quad \int d^4x\, C\, I_{\mathrm{inv}}(F, \text{matter}) \qquad (C \text{ abelian})$$

$$\text{III.} \quad \int d^4x\, \underbrace{(C\, A'_\mu - C' A_\mu)}_{\text{abelian}}\ \underbrace{j^\mu_{\mathrm{inv}}}_{\text{conserved current}} + \text{terms with antifields}$$

$$\Rightarrow \quad \mathcal{A} = \text{linear combination of type I.-III.} + s\,(\dots)$$

Do candidate anomalies of type II or III occur in $\mathcal{A}$?

Result: $\frac{d\mathcal{A}}{dg} = s\,\mathcal{B}_\star$ with $\mathcal{B}_\star$ an integrated star-polynomial (!):

$$\mathcal{B}_\star = \frac{\mathrm{i}\theta^{\alpha\beta}}{2}\int \mathrm{Tr}\,(\widehat{A}_\alpha \star \partial_\beta d\widehat{A} \star d\widehat{A} - \tfrac{1}{2}\, d\widehat{A}_\alpha \star \widehat{A}_\beta \star d\widehat{A} \star \widehat{A} + \tfrac{3}{2}\, d\widehat{A} \star d\widehat{A}_\alpha \star \widehat{A} \star \widehat{A}_\beta$$
$$-\tfrac{1}{2}\, d\widehat{A}_\alpha \star \widehat{A}_\beta \star \widehat{A} \star d\widehat{A} + \partial_\alpha \widehat{A}_\beta \star d\widehat{A} \star \widehat{A} \star \widehat{A} + \text{terms with 5 or 6 } \widehat{A}\text{'s})$$

This implies

$$\mathcal{A} = \int \mathrm{Tr}[Cd(AdA + \tfrac{1}{2}A^3)] + s\,\mathcal{B}[A,g], \qquad \mathcal{B}[A,g] = \int_0^g dg'\ \mathcal{B}_\star[\widehat{A}(A,g'),g']$$

- $\mathcal{A}$ is cohomologically equivalent to standard Bardeen anomaly
- No additional anomalies or anomaly cancellation conditions (at 1-loop)
- $\mathcal{B}$ is the counterterm that kills the θ-dependent would-be anomalies in $\mathcal{A}$
- $\mathcal{B}$ is determined only up to BRST-invariant contributions
- $\mathcal{B}$ is <u>not</u> an integrated star-polynomial (in contrast to $\mathcal{B}_\star$)
- SW map is essential also for anomaly issue!

Conclusions

- Existence of SW map reflects stability of YM gauge transformations under consistent deformations
- Systematic construction of SW map by BRST cohomological tools
- Ambiguities of SW map: gauge transformations, gauge covariant shifts of A_μ and Lie algebra automorphisms in enveloping algebra
- SW maps for x-dependent θ can be analysed by the same tools
- No gauge anomalies or anomaly cancellation conditions in addition to those of commutative models (only well-known chiral anomalies)
- SW map essential for anomalies
- Counterterm $\mathcal{B}$ that removes θ-dependence of anomalies is constructible in terms of $\widehat{A}_\mu \Rightarrow$ works for every choice of SW map