Gauge invariant interactions, conservation laws and local BRST cohomology

F. Brandt (MPI Leipzig)

- Consistent deformations of gauge theories
- Conservation laws of first and higher order
- Use of local BRST cohomology in this context
- Related developments (?)
- Relation to renormalization (?)

Refs.: G. Barnich, FB, M. Henneaux,
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Introduction

Construction and classification of gauge invariant interactions:
Given gauge transformations: what are the gauge invariant actions?
Given gauge invariant action: are there nontrivial consistent deformations of the action and gauge transformations?

2 types of nontrivial deformations:
type I: only action is nontrivially deformed
type II: both action and gauge transformations are nontrivially deformed

What one finds:

- Type II deformations are very often (always?) related to conservation laws
- Existence and structure of conservation laws of "higher order" are intimately related to gauge symmetries
- Local BRST cohomology is a powerful tool to study these topics

Remark: algebraic renormalizability (particularly in the "modern sense") depends decisively on absence of type II deformations

Consistent deformations

Consider an action $I^{(0)}[\varphi]$ with gauge invariance $\delta_\lambda^{(0)}$, i.e. $\delta_\lambda^{(0)} I^{(0)}[\varphi] = 0$.

Consistent deformations of $I^{(0)}[\varphi]$, $\delta_\lambda^{(0)}$:

$$I[\varphi, g] = I^{(0)}[\varphi] + \sum_{k \geq 1} g^k I^{(k)}[\varphi], \quad \delta_\lambda = \delta_\lambda^{(0)} + \sum_{k \geq 1} g^k \delta_\lambda^{(k)}, \quad \delta_\lambda I[\varphi, g] = 0$$

Trivial deformations: a deformation is called trivial if there are field redefinitions $\widehat{\varphi}(\varphi, g)$, $\widehat{\lambda}(\lambda, \varphi, g)$ such that

$$I[\widehat{\varphi}(\varphi, g), g] = I^{(0)}[\varphi], \quad (\delta_{\widehat{\lambda}} \widehat{\varphi})(\varphi, \lambda, g) \approx \delta_\lambda^{(0)} \widehat{\varphi}(\varphi, g)$$

Type I: $I \not\sim I^{(0)}$, $\delta \sim \delta^{(0)}$

Type II: $I \not\sim I^{(0)}$, $\delta \not\sim \delta^{(0)}$

Examples:

All familiar gauge invariant actions with free parameters;
deformation parameters $\equiv$ free parameters (gauge coupling constants, masses ...)

Pure YM theory:

$$I = -\frac{1}{4}\int d^n x \, \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$$

$$\delta_\lambda A_\mu = \partial_\mu \lambda + g[A_\mu, \lambda]$$

$$I^{(0)} = -\int d^n x \, \mathrm{Tr}\,(\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]}), \qquad \delta_\lambda^{(0)} A_\mu = \partial_\mu \lambda$$

$$I^{(1)} = -\int d^n x \, \mathrm{Tr}\,([A_\mu, A_\nu]\partial^\mu A^\nu), \qquad \delta_\lambda^{(1)} A_\mu = [A_\mu, \lambda]$$

$$I^{(2)} = -\frac{1}{4}\int d^n x \, \mathrm{Tr}\,([A_\mu, A_\nu]\,[A^\mu, A^\nu]), \qquad \delta_\lambda^{(2)} A_\mu = 0$$

Pure YM theory is a type II deformation of a free Maxwell-type theory

Result on deformations of YM theories

Pure YM theories (including eff. theories):

- Semisimple gauge group:
  only consistent deformations of type I (no nontrivial deformations of gauge transformations at all!)

- Gauge group with abelian factors, esp. free theories (1st order result):

$$I^{(1)} \sim I^{(1)}_{\text{inv}} + \int d^n x \, (\underbrace{k_i^{\Delta} A^i_\mu j^\mu_\Delta}_{\text{Noether type}} - \underbrace{\tfrac{1}{2} f_{ijk} F^{\mu\nu i} A^j_\mu A^k_\nu}_{\text{YM type}})$$

$$+ \int d^n x \, \underbrace{k_{ijk} [-\tfrac{1}{2} F^{\mu\nu i} A^j_\mu A^k_\nu + \tfrac{2}{n-4} \, x^\mu (F^i_{\mu\rho} F^{\nu\rho j} - \tfrac{1}{4} \, \delta^\nu_\mu F^i_{\rho\sigma} F^{\rho\sigma j}) A^k_\nu]}_{\text{peculiar type } (n \neq 4)}$$

  where

  $I^{(1)}_{\text{inv}}$: $\int$ G-inv. polynomials in $F, DF, \ldots$ plus Chern-Simons terms in odd dims.
  A^i_μ : abelian gauge fields
  j^μ_Δ : gauge invariant conserved currents of the undeformed theory
  $k_i^\Delta, f_{ijk}, k_{ijk}$: constants with $f_{ijk} = f_{[ijk]}$, $k_{ijk} = -k_{ikj} = k_{jik}$

Corresponding first order deformations of gauge transformations:

$$\delta_\lambda^{(1)} A_\mu^a \sim k_i^{\Delta}\, \lambda^i G^a_{\Delta\mu} \quad \text{if } A_\mu^a \text{ nonabelian}$$

$$\delta_\lambda^{(1)} A_\mu^i \sim \underbrace{k_j^{\Delta}\, \lambda^j G^i_{\Delta\mu}}_{\text{Noether type}} + \underbrace{f_{ijk} A_\mu^j \lambda^k}_{\text{YM type}} + \underbrace{k_{ijk}(A_\mu^j + \tfrac{2}{n-4}\, x^\nu F^j_{\nu\mu})\lambda^k}_{\text{peculiar type } (n\neq 4)}$$

where $G^a_{\Delta\mu}$ is the (infinitesimal) transformation of A_μ^a under the global symmetry of $I^{(0)}$ that corresponds to j^μ_Δ:

$$G^a_{\Delta\mu} \frac{\delta I^{(0)}}{\delta A_\mu^a} = \partial_\mu j^\mu_\Delta$$

Remarks:

- Noether type deformations involve conserved currents j^μ ("conservation laws of first order") via $j^\mu A_\mu^i$
- YM type and peculiar deformations involve "conservation laws of second order" $j^{\mu\nu} = -j^{\nu\mu}$ via $j^{\mu\nu} A_\mu^i A_\nu^j$ where $j^{\mu\nu} = F^{\mu\nu k}$
- Higher orders $\longrightarrow$ relations between the coefficients, e.g. $\sum_m f_{m[ij} f_{k]ml} = 0$
- From point of view of free theories: nonabelian YM transformations are already deformations – further deformation impossible

Conservation laws of first and higher order

Conserved current j^μ:

$$\partial_\mu j^\mu(\varphi) \approx 0 \ :\Leftrightarrow\ \partial_\mu j^\mu(\varphi) = G^i(\varphi, \partial)\, \frac{\delta I[\varphi]}{\delta \varphi^i}$$

In differential form notation:

$$d\, j^{n-1} \approx 0 \quad (d = dx^\mu \partial_\mu,\ \ j^{n-1} = \tfrac{1}{(n-1)!}\, dx^{\mu_1} \dots dx^{\mu_{n-1}} \varepsilon_{\mu_1 \dots \mu_n}\, j^{\mu_n})$$

Generalization ("conservation law of order p"):

$$d\, j^{n-p} \approx 0 \quad \Leftrightarrow \quad \partial_{\mu_1} j^{\mu_1 \cdots \mu_p} \approx 0 \quad (j^{\mu_1 \cdots \mu_p} = j^{[\mu_1 \cdots \mu_p]})$$

Trivial conservation law: $j^{n-p} \approx d\, \omega^{n-p-1}$

Characteristic cohomology $H^p_{\text{char}}(d)$ (of the field equations): solutions of

$$d\, j^p \approx 0 \text{ with equivalence relation } j^p \sim \tilde{j}^p \ :\Leftrightarrow\ j^p - \tilde{j}^p \approx d\, \omega^{p-1}$$

Remark: $H^p_{\text{char}}(d)$ is completely different from de Rham cohomology:

1. d is not $dx^\mu \dfrac{\partial}{\partial x^\mu}$ but $d = dx^\mu \partial_\mu$ with $\partial_\mu = \dfrac{\partial}{\partial x^\mu} + \partial_\mu \varphi \dfrac{\partial}{\partial \varphi} + \partial_\mu \partial_\nu \varphi \dfrac{\partial}{\partial(\partial_\nu \varphi)} + \dots$
2. $H^p_{\text{char}}(d)$ is a cohomology "on-shell"

General result on conservation laws

“Normal” theories of reducibility order r ($r = -1$ for theories without gauge symmetry; $r = 0$ for irreducible gauge theories such as YM theory and GR) can have nontrivial conservation laws only up to order $r + 2$:

$$H^p_{\text{char}}(d) = 0 \quad \text{for} \quad 0 < p < n - r - 2 \;;$$

$$d\,j^{n-p} \approx 0, \quad n > p > r + 2 \quad \Rightarrow \quad j^{n-p} \approx d\,\omega^{n-p-1}$$

Hence:

- Non-gauge theories can only have nontrivial conservation laws of 1st order
- Irreducible gauge theories can only have nontrivial conservation laws of 1st and 2nd order

Pure YM theory:

2nd order conservation laws are exhausted by dual field strengths of abelian gauge fields ($j^{\mu\nu} \sim k_i F^{\mu\nu i}$)

no 2nd order conservation law when gauge group is semisimple

Field-antifield formalism, BRST-differential (for Lagrangean theories)

Fields: $\{\phi\} = \{\varphi, \underbrace{C, \ldots}_{\text{if gauge symm is reducible}}\} \overset{\text{paired}}{\longleftrightarrow}$ antifields: $\{\phi^*\} = \{\varphi^*, C^*, \ldots\}$

Antibracket: $(F,G) = \int d^n x \, F \Big(\frac{\overleftarrow{\delta}}{\delta\phi} \frac{\overrightarrow{\delta}}{\delta\phi^*} - \frac{\overleftarrow{\delta}}{\delta\phi^*} \frac{\overrightarrow{\delta}}{\delta\phi} \Big) G$

Master action:

$$S[\phi,\phi^*] = \underbrace{I[\varphi]}_{\text{ordinary action}} + \underbrace{\int d^n x \, \varphi^* \delta_C \varphi}_{\text{gauge transformations}} + \underbrace{\ldots}_{\text{gauge algebra}} \qquad \text{such that} \qquad \underbrace{(S,S) = 0}_{\text{master equation}}$$

BRST differential (coboundary operator):

$$\boxed{s = (S, \,\cdot\,)} \qquad (\Rightarrow \; s^2 = 0)$$

contains complete information about:

- gauge symmetry (transformations, algebra, . . .)
- field eqs. (eqs. themselves, Noether identities, . . .)

$$s\,\phi = (S,\phi) = -\frac{\delta^R S}{\delta\phi^*}; \qquad s\,\varphi = \pm\delta_C\varphi + \ldots \qquad \text{contains gauge transformations}$$

$$s\,\phi^* = (S,\phi^*) = \frac{\delta^R S}{\delta\phi}; \qquad s\,\varphi^* = \pm\frac{\delta I}{\delta\varphi} + \ldots \qquad \text{contains (lhs of) eqs. of motion}$$

Ghost number (gh) and antifield number (af):

	φ	φ^*	C	C^*	$\ldots$
gh	0	-1	1	-2	
af	0	1	0	2	

$$\mathrm{gh}(s) = \mathrm{gh}((\ ,\ )) = 1$$

Expansion of s in antifield number:

$$s = \delta + \gamma + s_1 + \ldots, \quad \mathrm{af}(\delta) = -1, \quad \mathrm{af}(\gamma) = 0, \quad \mathrm{af}(s_1) = 1, \quad \ldots$$

δ : "Koszul-Tate differential" $\longrightarrow$ field equations

γ : "exterior derivative along gauge orbits" $\longrightarrow$ gauge symmetry

Example: pure YM theory

$$S = -\frac{1}{4}\int d^n x \, \mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu} + A^{*\mu}(\partial_\mu C + [A_\mu, C]) + C^*CC]$$

$$sA_\mu = \partial_\mu C + [A_\mu, C], \quad s\,C = -CC$$

$$sA^{*\mu} = D_\nu F^{\nu\mu} - \{C, A^{*\mu}\}, \quad s\,C^* = -D_\mu A^{*\mu} - [C, C^*]$$

$$s = \delta + \gamma$$

$$\delta A_\mu = \delta C = 0, \quad \delta A^{*\mu} = D_\nu F^{\nu\mu}, \quad \delta C^* = -D_\mu A^{*\mu}$$

$$\gamma A_\mu = \partial_\mu C + [A_\mu, C], \quad \gamma C = -CC, \quad \gamma A^{*\mu} = -\{C, A^{*\mu}\}, \quad \gamma C^* = -[C, C^*]$$

Local BRST cohomology

Defined in the space of local p-forms with ghost number g:

$$\omega^{g,p}(\phi,\phi^*) = \frac{1}{p!}\underbrace{dx^{\mu_1}\dots dx^{\mu_p}}_{\text{wedge product}}\underbrace{f_{\mu_1\dots\mu_p}(x,\phi,\phi^*,\partial\phi,\partial\phi^*,\dots)}_{\text{ghost number } g}$$

Local BRST cohomology $H^{g,p}(s|d)$: solutions $\omega^{g,p}$ of

$$s\,\omega^{g,p} + d\,\omega^{g+1,p-1} = 0$$

with equivalence relation ($\sim$)

$$\omega^{g,p} \sim \omega^{g,p} + s\,\omega^{g-1,p} + d\,\omega^{g,p-1}$$

Analogously $H^p_k(\delta|d)$: local p-forms ω^p_k with antifield number k satisfying

$$\delta\,\omega^p_k + d\,\omega^{p-1}_{k-1} = 0, \quad \omega^p_k \sim \omega^p_k + \delta\,\omega^p_{k+1} + d\,\omega^{p-1}_k$$

Consistent deformations and local BRST-cohomology

$$S = S^{(0)} + \sum_{k\geq 1} g^k S^{(k)}, \quad (S,S) = 0$$

Expansion of master equation in g:

$$(S^{(0)}, S^{(1)}) = 0, \ \ (S^{(1)}, S^{(1)}) + 2(S^{(0)}, S^{(2)}) = 0, \quad \ldots$$

$$\Leftrightarrow \quad s^{(0)} S^{(1)} = 0, \quad (S^{(1)}, S^{(1)}) = -2 s^{(0)} S^{(2)}, \quad \ldots$$

where $s^{(0)} S^{(1)} = 0 \ :\Leftrightarrow \ s^{(0)}\omega^{0,n} + d\omega^{1,n-1} = 0$ with $S^{(1)} = \int \omega^{0,n}$

Trivial deformations: anticanonical transforms. $\widehat{\phi}(\phi,\phi^*,g)$, $\widehat{\phi}^*(\phi,\phi^*,g)$ s.t.

$$S[\widehat{\phi}(\phi,\phi^*,g), \widehat{\phi}^*(\phi,\phi^*,g), g] = S^{(0)}(\phi,\phi^*)$$

$$\stackrel{\frac{d}{dg}}{\Rightarrow} \quad \frac{\partial S}{\partial g} - \underbrace{(S,\Xi)}_{s\,\Xi} = 0, \quad \frac{d\widehat{\phi}}{dg} = (\Xi, \widehat{\phi}), \quad \frac{d\widehat{\phi}^*}{dg} = (\Xi, \widehat{\phi}^*)$$

$$\Rightarrow \quad S^{(1)} = (S^{(0)}, \Xi^{(0)}) = s^{(0)}\,\Xi^{(0)}, \quad \ldots$$

Consistent deformations are determined by $H^{0,n}(s^{(0)}|d)$ and $H^{1,n}(s^{(0)}|d)$

Conservation laws and local BRST-cohomology

$$d\,j^p(\varphi)\approx 0 \quad\Leftrightarrow\quad d\,j^p = G^i_{\mu_1\dots\mu_{p+1}}(\varphi,\partial)\,\frac{\delta I[\varphi]}{\delta\varphi^i}\,dx^{\mu_1}\dots dx^{\mu_{p+1}}$$

$$\Leftrightarrow\quad d\,j^p+\delta\,\omega^{p+1}_1=0 \quad\text{with}\quad \omega^{p+1}_1=\pm G^i_{\mu_1\dots\mu_{p+1}}(\varphi,\partial)\,\varphi^*_i\,dx^{\mu_1}\dots dx^{\mu_{p+1}}$$

$$\Rightarrow\quad H^p_{\text{char}}(d)\simeq H^{p+1}_1(\delta|d)\quad\text{for}\quad 0<p<n$$

First Noether theorem: $\underbrace{H^n_1(\delta|d)}_{\substack{\text{global symms.}\\ \omega^{1,n}=\varphi^*_i G^i(\varphi)d^nx}} \simeq \underbrace{\begin{cases} H^{n-1}_{\text{char}}(d) & \text{if } n>1\\ H^0_{\text{char}}(d)/\{\text{constants}\} & \text{if } n=1\end{cases}}_{\text{conserved currents}}$

Generalization to conservation laws of arbitrary order:

$$0<k<n:\quad H^n_k(\delta|d)\simeq H^{n-1}_{k-1}(\delta|d)\simeq\dots\simeq H^{n-k+1}_1(\delta|d)\simeq H^{n-k}_{\text{char}}(d)$$

$$k=n:\quad H^n_n(\delta|d)\simeq H^{n-1}_{n-1}(\delta|d)\simeq\dots\simeq H^1_1(\delta|d)\simeq H^0_{\text{char}}(d)/\{\text{constants}\}$$

$$k>n:\quad H^n_k(\delta|d)\simeq H^{n-1}_{k-1}(\delta|d)\simeq\dots\simeq H^1_{k-n+1}(\delta|d)\simeq H^0_{k-n}(\delta)=0$$

Furthermore $H^{-k,p}(s|d)\simeq H^p_k(\delta|d)$ for $k>0$

Local BRST cohomology in negative ghost numbers $\simeq$ conservation laws

General result: $H^n_k(\delta|d)=0$ for $k>r+2$ ($\Rightarrow H^p_{\text{char}}(d)=0$ for $p<n-r-2$)

Relation between consistent deformations and conservation laws in pure YM

$$s^{(0)}\,S^{(1)}=0,\qquad S^{(1)}=\int(\omega_0^{0,n}+\omega_1^{0,n}+\ldots+\omega_k^{0,n})$$

One finds (modulo trivial contributions):

$$\omega_k^{0,n}=\sum\underbrace{\omega_k^n(A,A^*,C^*)}_{\in H_k^n(\delta|d)}\,P(C)$$

Results on $H(\delta|d)$:

$H_k^n(\delta|d)=0$ for $k>2$, $H_2^n(\delta|d)=\{C_i^*\}$, $H_1^n(\delta|d)=\{A_a^{*\mu}G_\mu^a(A)d^nx\}$; leads to

$k>2:\quad \omega_k^{0,n}=0\quad$ (without loss of generality)

$k=2:\quad \omega_2^{0,n}=\frac{1}{2}\kappa_{ijk}\,C_i^*C^jC^k\,d^nx\quad$ where $\quad\kappa_{ijk}=-\kappa_{ikj}=\text{constant}$

$\kappa_{[ijk]}=f_{ijk}\ \longrightarrow$ YM type deformations

$\kappa_{(ij)k}=k_{ijk}\ \longrightarrow$ peculiar deformations

$k=1:\quad \omega_1^{0,n}=\sum A_a^{*\mu}G_\mu^a(A)C^i\,d^nx\ \longrightarrow$ Noether type deformations

$k=0:\quad \int\omega_0^{0,n}=I_{\text{inv}}$

Renormalization vs. consistent deformations

Extended effective action:

$$\Gamma = S|_{\text{gauge fixed}} + \sum_k \hbar^k \Gamma^{(k)}$$

ST identity:

$$(\Gamma, \Gamma) = 0$$

$$\Rightarrow \ (S, \Gamma^{(1)}_{\text{counter}}) = 0 \ \longleftarrow \ H^{0,n}(s|d)$$

Anomalies:

$$(\Gamma, \Gamma) = \mathcal{A} \ \longleftarrow \ H^{1,n}(s|d)$$

Extended action:

$$S = S^{(0)} + \sum_k g^k S^{(k)}$$

Master equation:

$$(S, S) = 0$$

$$\Rightarrow \ (S^{(0)}, S^{(1)}) = 0 \ \longleftarrow \ H^{0,n}(s^{(0)}|d)$$

Obstructions at higher order:

$$(\sum_k S^{(k)}, \sum_k S^{(k)}) = A \ \longleftarrow \ H^{1,n}(s^{(0)}|d)$$

Related developments

- Extended field-antifield formalism [F.B., M. Henneaux, A. Wilch '97/'98]

  Extended BRST differential that includes global symmetries
  Consistent deformations of gauge and global symmetries

- Asymptotic conservation laws [G. Barnich, FB '01]

  Classification and construction of 2nd order asymptotic conservation laws
  Analog of Noether theorem for these conservation laws
  Charges for gauge symmetries

- Aspects of non-commutative gauge theories
  [G. Barnich, FB, M. Grigoriev '02/'03]

  Non-commutative gauge theories as consistent deformations of commutative theories
  Existence, construction, ambiguities of Seiberg-Witten maps

Conclusion

- Consistent deformations determined by $H^{0,n}(s^{(0)}|d)$ and $H^{1,n}(s^{(0)}|d)$
- Conservation laws determined by $H^{g,n}(s^{(0)}|d)$ with $g < 0$
- Antifield dependent BRST cohomology is decisive
- Nontrivial deformations of gauge symmetries are intimately related to conservation laws
- Nontrivial conservation laws of higher order exist only in gauge theories
- General theorems and results on (potentially) relevant theories available (YM, GR, susy and sugra theories, theories with p-form gauge potentials, string theories)

Extended field-antifield formalism

[FB, M. Henneaux, A. Wilch '97/'98]

Inclusion of global symmetries:

$$S_{\text{ext}}[\phi,\phi^*,\underbrace{\xi\,,\xi^*}_{\substack{\text{constant ghosts and antifields}\\ \text{for global symmetries}}}] = S[\phi,\phi^*] + \underbrace{\int d^nx\,\varphi^*\delta_\xi\varphi}_{\substack{\text{global symmetry}\\ \text{transformations}}} + \dots$$

Extended BRST differential:

$$(S_{\text{ext}}\,,S_{\text{ext}})_{\text{ext}} = 0,\ \ s_{\text{ext}} = (S_{\text{ext}}\,,\ \cdot\ )_{\text{ext}}\,,\ \ s_{\text{ext}}^2 = 0$$

Allows one to analyse consistent deformations of gauge and global symmetries

Asymptotic conservation laws [G. Barnich, FB '01]

Asymptotic cons. law of order p: $d\,j^{n-p}_{\text{asymp}} \stackrel{\approx}{\to} 0$; trivial if $j^{n-p}_{\text{asymp}} \stackrel{\approx}{\to} d\,\omega^{n-p-1}$

Result for irreducible gauge theories (under suitable assumptions on asymptotics):
$j^{n-2}_{\text{asymp}} \longleftrightarrow$ asymptotic gauge symmetries ("asymptotic Killing vectors");
when field equations at most of 2nd order:

$$j^{n-2}_{\text{asymp}} \simeq (d^{n-2}x)_{\mu\nu}\Big[\varphi^i\,\frac{\partial^S s^\nu(\varphi,\varepsilon)}{\partial(\partial_\mu\varphi^i)} + \Big(\frac{4}{3}\,\partial_\rho\varphi^i - \frac{2}{3}\,\varphi^i\partial_\rho\Big)\,\frac{\partial^S s^\nu(\varphi,\varepsilon)}{\partial(\partial_\rho\partial_\mu\varphi^i)}\Big]$$

where $s^\nu(\varphi,\varepsilon)$ is directly determined by the Lagrangian and gauge symmetries, and involves the parameters $\varepsilon(x)$ of asymptotic gauge symmetries.

Remarks:

- Noether type theorem for 2nd order asymptotic conservation laws
- $Q = \int j^{n-2}_{\text{asymp}}$ can be interpreted as charge of gauge symmetries, e.g.:
  Electric charge in electrodynamics:
  $$Q = \int_{\partial\Sigma} d\sigma_i\, F^{0i} \quad (\varepsilon = \text{constant})$$
  Energy in GR for asymptotically flat spacetime (ADM mass formula):
  $$E = \int_{\partial\Sigma} d\sigma_i\, \eta^{ik}\eta^{jl}(\partial_j g_{kl} - \partial_k g_{jl}) \quad (\varepsilon = \text{constant})$$

Non-commutative gauge theories

Weyl-Moyal product:

$$f_1 \star f_2 = f_1 \, \exp(\overleftarrow{\partial_\mu} \, \tfrac{\mathrm{i}}{2} \, g \, \theta^{\mu\nu} \, \overrightarrow{\partial}_\nu) \, f_2, \qquad \theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constant}$$

Action and gauge transformations:

$$\widehat{I}[\widehat{A}] = -\tfrac{1}{4} \int d^n x \, \mathrm{Tr} \, (\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}), \qquad \widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu + [\widehat{A}_\mu \overset{\star}{,} \widehat{A}_\nu]$$

$$\widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu = \partial_\mu \widehat{\lambda} + [\widehat{A}_\mu \overset{\star}{,} \widehat{\lambda}]$$

Seiberg-Witten map [Seiberg, Witten 1999]:

$$\widehat{A}_\mu = \widehat{A}_\mu(A) = A_\mu + \tfrac{\mathrm{i}}{4} \, g \, \theta^{\alpha\beta} \{F_{\alpha\mu} + \partial_\alpha A_\mu, A_\beta\} + \ldots$$

$$\widehat{\lambda} = \widehat{\lambda}(\lambda, A) = \lambda - \tfrac{\mathrm{i}}{4} \, g \, \theta^{\alpha\beta} \{A_\alpha, \partial_\beta \lambda\} + \ldots$$

such that

$$\delta_\lambda A_\mu = \partial_\mu \lambda + [A_\mu, \lambda] \;\Rightarrow\; \delta_\lambda \widehat{A}_\mu(A) = (\widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu)(A, \lambda)$$

$$\begin{aligned} I[A] = \widehat{I}[\widehat{A}(A)] \;&=\; -\tfrac{1}{4} \int d^n x \, \mathrm{Tr} \, (F_{\mu\nu} F^{\mu\nu}) \\ &\quad + \mathrm{i} \, g \, \theta^{\alpha\beta} \int d^n x \, \mathrm{Tr} \, (\tfrac{1}{8} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} F_{\alpha\mu} F_{\beta\nu} F^{\mu\nu}) + \ldots \end{aligned}$$

Noncommutative $U(N)$ gauge theory is a type I deformation of ordinary (commutative) YM theory with gauge group $U(N)$