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Asymptotic symmetries, conservation laws and central charges

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A motivation:

- Meaningful charges for gauge symmetries (electric charge in ED, energy or momentum in GR...)

Results:

- Correspondence between asympt. reducibility parameters and asympt. conserved $n-2$ forms
- Universal formula for $n-2$ forms and charges
- Lie algebra associated with charges
- Central charges: formula, cocycle property
- Example: Einstein gravity

The Noether charge puzzle for gauge symmetries

Gauge invariance:

$$\delta_\varepsilon \phi^i \frac{\hat{\partial} L}{\hat{\partial}\phi^i} = \partial_\mu j^\mu_\varepsilon, \qquad \delta_\varepsilon \phi^i = R^i_\alpha(x,\phi,\partial)\varepsilon^\alpha$$

Using $(\partial\varepsilon)X = \partial(\varepsilon X) - \varepsilon(\partial X)$ ("integration by parts"):

$$\delta_\varepsilon \phi^i \frac{\hat{\partial} L}{\hat{\partial}\phi^i} = \varepsilon^\alpha \underbrace{R^{i+}_\alpha(x,\phi,\partial)\frac{\hat{\partial} L}{\hat{\partial}\phi^i}}_{\substack{=0 \\ \text{(Noether identity)}}} + \partial_\mu S^\mu_\varepsilon = \partial_\mu S^\mu_\varepsilon$$

S^μ_ε is on-shell vanishing Noether current:

$$\boxed{S^\mu_\varepsilon = S^{\mu i}(x,\varepsilon,\phi,\partial)\frac{\hat{\partial} L}{\hat{\partial}\phi^i} \approx 0}$$

Other currents: $\partial_\mu(j^\mu_\varepsilon - S^\mu_\varepsilon) = 0 \;\Rightarrow$

$$\boxed{j^\mu_\varepsilon = S^\mu_\varepsilon + \partial_\nu k^{\mu\nu}_\varepsilon \approx \partial_\nu k^{\mu\nu}_\varepsilon, \qquad k^{\nu\mu}_\varepsilon = -k^{\mu\nu}_\varepsilon}$$

for some "superpotential" $k^{\mu\nu}_\varepsilon$. Noether charge:

$$Q = \int_\Sigma (d^{n-1}x)_\mu \, j^\mu_\varepsilon \approx \int_{\partial\Sigma} k_\varepsilon, \qquad k_\varepsilon = (d^{n-2}x)_{\mu\nu}\, k^{\mu\nu}_\varepsilon$$

Q is surface integral of an $n-2$ form k_ε.

This suggests to derive gauge charges from superpotentials rather than from currents. But:

- Which k's are meaningful?
- For which gauge parameters ε^α?
- Role of boundary conditions at $\partial\Sigma$?

Example: ED

$$L = \mathrm{i}\bar{\psi}\not{D}\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad D_\mu\psi = (\partial_\mu - \mathrm{i}eA_\mu)\psi$$

$$\delta_\varepsilon A_\mu = \partial_\mu\varepsilon, \qquad \delta_\varepsilon\psi = \mathrm{i}e\varepsilon\psi$$

Standard derivation of "the" e.m. current:

$$\varepsilon = 1 \ \Rightarrow\ \delta_1\phi^i\,\frac{\hat{\partial}L}{\hat{\partial}\phi^i} = \partial_\mu j_1^\mu, \qquad j_1^\mu = e\bar{\psi}\gamma^\mu\psi$$

One has:

$$\frac{\hat{\partial}L}{\hat{\partial}A_\mu} = \partial_\nu F^{\nu\mu} + e\bar{\psi}\gamma^\mu\psi$$

and thus

$$j_1^\mu = \underbrace{\frac{\hat{\partial}L}{\hat{\partial}A_\mu}}_{S_1^\mu} + \partial_\nu\underbrace{F^{\mu\nu}}_{k_1^{\mu\nu}} \approx \partial_\nu F^{\mu\nu}$$

i.e., $F^{\mu\nu}$ is a superpotential for this particular choice

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More generally, for arbitrary $\varepsilon = \varepsilon(x)$:

$$\delta_\varepsilon\phi^i\,\frac{\hat{\partial}L}{\hat{\partial}\phi^i} = \partial_\mu(\underbrace{\varepsilon\partial_\nu F^{\nu\mu} + e\varepsilon\bar{\psi}\gamma^\mu\psi}_{S_\varepsilon^\mu = \varepsilon\frac{\hat{\partial}L}{\hat{\partial}A_\mu}\approx 0})$$

Description of asymptotics

$$\phi^i = \bar{\phi}^i(x) + \varphi^i$$

with $\bar{\phi}^i$ a background solution,

$$\frac{\hat{\partial} L}{\hat{\partial} \phi^i}\bigg|_{\bar{\phi}(x)} = 0,$$

and φ^i bounded by some $\chi^i(x)$,

$$\varphi^i \longrightarrow O(\chi^i(x)), \quad \chi^i(x)/\bar{\phi}^i(x) \longrightarrow 0.$$

Expansion in φ's:

$$L(x,\phi) = \sum_r L^{(r)}(x,\varphi,\bar{\phi}^i(x)), \quad r\text{: degree in } \varphi\text{'s}$$

Basic assumption: asymptotic linearizability,

$$\frac{\hat{\partial} L}{\hat{\partial} \phi^i} = \frac{\hat{\partial} L^{(2)}}{\hat{\partial} \varphi^i} + \text{asymptotically irrelevant terms}$$

Notice: the φ's are not "small". In the bulk they can be large, near the boundary they are small as compared to $\bar{\phi}$'s but do not necessarily tend to zero.

Asymptotic reducibility parameters (ARPs)

- Let $\chi_i(x)$ characterize asymptotics of field eqs.,

$$\forall \varphi^i : \quad d^n x \, \frac{\hat{\partial} L^{(2)}}{\hat{\partial} \varphi^i} \longrightarrow O(\chi_i).$$

  ARPs are "parameters" $f^\alpha(x)$ satisfying

$$\boxed{\forall \psi_i \longrightarrow O(\chi_i) : \quad \psi_i R^i_\alpha(x, \bar{\phi}, \partial) \, f^\alpha \longrightarrow 0} \qquad (1)$$

- Notice: the solutions of

$$R^i_\alpha(x, \bar{\phi}, \partial) \, f^\alpha(x) = 0$$

  are ARPs. These are the "Killing vectors of the background". Hence:
  ARPs may be interpreted as "asymptotic Killing vectors of the background" satisfying

$$R^i_\alpha(x, \bar{\phi}, \partial) \, f^\alpha(x) \longrightarrow o(1/\chi_i)$$

- Remark: in general, gauge transformations whose parameters are ARPs need <u>not</u> preserve the BCs of the fields!

- ARPs satisfying (1) just because they fall off fast enough at the boundary are called trivial. They are characterized by $f^\alpha \longrightarrow o(\chi^\alpha)$ for some $\chi^\alpha(x)$.
  Equivalence classes of ARPs:

$$f^\alpha \sim f'^\alpha \;:\Leftrightarrow\; f^\alpha - f'^\alpha \longrightarrow o(\chi^\alpha)$$

Asymptotically conserved $n-2$ forms

- These are $n-2$ forms $k = (dx^{n-2})_{\mu\nu}\, k^{\mu\nu}(x,\varphi)$ which are linear in the φ's and satisfy

$$\boxed{\forall \varphi^i : \qquad dk \longrightarrow (d^{n-1}x)_\mu\, s^{\mu i}(x,\partial)\, \frac{\hat{\partial} L^{(2)}}{\hat{\partial} \varphi^i}} \qquad (2)$$

for some operators $s^{\mu i}(x,\partial)$. Notation:

$$(dx^{n-p})_{\mu_1\dots\mu_p} = \tfrac{1}{p!(n-p)!}\, \epsilon_{\mu_1\dots\mu_n} dx^{\mu_{p+1}} \dots dx^{\mu_n}$$

- Trivial asymptotically conserved $n-2$ forms:

$$k \longrightarrow (dx^{n-2})_{\mu\nu}\, t^{\mu\nu}(x,\partial)\, \frac{\hat{\partial} L^{(2)}}{\hat{\partial} \varphi^i} + du$$

for some operators $t^{\mu\nu}(x,\partial)$ and $n-3$ form u.

- Equivalence classes:

$$k \sim k' \;\; :\Leftrightarrow \;\; k - k' \longrightarrow (dx^{n-2})_{\mu\nu}\, t^{\mu\nu}(x,\partial)\, \frac{\hat{\partial} L^{(2)}}{\hat{\partial} \varphi^i} + du$$

- Notice: (2) is the asymptotic analog for $n-2$ forms of the conservation law for currents.

Correspondence between k's and f's

121-correspondence between equivalence classes:

$$[\,f\,] \longleftrightarrow [\,k_f\,]$$

given by

$$\boxed{k_f = \varrho_\varphi\, s_f, \qquad s_f = S^{\mu i}(x, f, \bar{\phi}, \partial)\, \frac{\hat{\partial} L^{(2)}}{\hat{\partial}\varphi^i}\, (d^{n-1}x)_\mu}$$

with ϱ_φ the homotopy operator for d.

E.g., when s_f contains at most second order derivatives of φ's (standard case), this yields explicitly:

$$\boxed{k_f^{\mu\nu} = \frac{1}{2}\varphi^i\, \frac{\partial^S s_f^\nu}{\partial \varphi^i_\mu} + \Big(\frac{2}{3}\varphi^i_\lambda - \frac{1}{3}\varphi^i\partial_\lambda\Big)\frac{\partial^S s_f^\nu}{\partial\varphi^i_{\lambda\mu}} - (\mu \leftrightarrow \nu)}$$

where

$$\varphi^i_{\mu_1\dots\mu_r} = \partial_{\mu_1}\dots\partial_{\mu_r}\varphi^i, \qquad \frac{\partial^S \varphi^i_{\mu_1\dots\mu_r}}{\partial\varphi^j_{\nu_1\dots\nu_r}} = \delta^i_j\, \delta^{\nu_1}_{(\mu_1}\dots\delta^{\nu_r}_{\mu_r)}$$

Charges:

$$Q_f = \int_{\partial\Sigma} k_f(x, \varphi, \bar{\phi}) + N_f$$

with N_f the charge of the background (a priori arbitrary; normalization)

- Introduce antifields φ^*_i, C^*_α with

$$\varphi^*_i d^n x \longrightarrow O(\chi_i), \quad C^*_\alpha d^n x \longrightarrow O(R^{i+}_\alpha(x,\bar\phi,\partial)\,\chi_i)$$

and Koszul-Tate differential δ for $L^{(2)}$:

$$\delta C^*_\alpha = R^{i+}_\alpha(x,\bar\phi,\partial)\,\varphi^*_i, \quad \delta\varphi^*_i = \frac{\widehat{\partial} L^{(2)}}{\widehat{\partial}\varphi^i}, \quad \delta\phi^i = 0.$$

- The condition $\psi_i R^i_\alpha(x,\bar\phi,\partial) f^\alpha \longrightarrow 0$ turns into

$$\boxed{\delta\omega^n + d\omega^{n-1} \longrightarrow 0, \quad \omega^n = C^*_\alpha\, f^\alpha(x)\, d^n x,} \tag{3}$$

and the condition $dk \longrightarrow s^{\mu i}\,\frac{\widehat{\partial} L^{(2)}}{\widehat{\partial}\varphi^i}\,(d^{n-1}x)_\mu$ into

$$\boxed{\delta\omega^{n-1} + dk \longrightarrow 0,} \tag{4}$$

for some ω^{n-1}, respectively.

- Proof of $[\,f\,] \longleftrightarrow [\,k_f\,]$:
(3) $\Leftrightarrow$ (4) with $f \sim 0 \Leftrightarrow k \sim 0$, implied by cohomological properties of d and δ (BCs and space of functions for φ's and f's must support these properties!)

- Formula for k_f:
derives from the fact that ω^{n-1} turns out to be

$$\omega^{n-1} \sim S^{\mu i}(x, f, \bar\phi, \partial)\,\varphi^*_i\,(d^{n-1}x)_\mu$$

Algebra and central charges

- Commutator algebra of gauge transformations:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi^i \approx R^i_\alpha(x,\phi,\partial)\, C^\alpha(x,\phi,\varepsilon_1,\varepsilon_2)$$

- The bracket of ARPs defined by

$$[f_1,f_2]^\alpha := C^\alpha(x,\bar\phi,f_1,f_2)$$

  establishes a Lie algebra $\mathfrak{g}$ at the level of ARPs and their equivalence classes because it can be shown that $C^\alpha(x,\phi,f_1,f_2)$ are again ARPs (under appropriate assumptions on BCs).

- Induced algebra on charges:

$$[Q_{f_1},Q_{f_2}] := \delta_{f_1}Q_{f_2} = Q_{[f_1,f_2]} - N_{[f_1,f_2]} + Z_{f_1,f_2}$$

  with Z's which are given by

$$Z_{f_1,f_2} = \int_{\partial\Sigma} k_{f_2}(x, R^i_\alpha(x,\bar\phi,\partial)f_1^\alpha,\bar\phi)$$

  and represent 2-cocycles on $\mathfrak{g}$:

$$Z_{f_1,f_2} = -Z_{f_2,f_1} \qquad ,$$
$$Z_{[f_1,f_2],f_3} + Z_{[f_2,f_3],f_1} + Z_{[f_3,f_1],f_2} = 0.$$

  At the level of the charges, $\mathfrak{g}$ may thus change to some $\mathfrak{g}'$ differing from $\mathfrak{g}$ by central extensions

- Remarks:

  a) $\mathfrak{g}$ is <u>not</u> related to the Lie algebra of the gauge symmetries of $L^{(2)}$ (the latter is Abelian, $\mathfrak{g}$ is in general non-Abelian).

  b) The proof of the cocycle property is nontrivial and uses properties of the homotopy operator ϱ_φ.

  c) The fact that the Z's are cocycles on $\mathfrak{g}$ allows one to determine the possible central charges from the Lie algebra cohomology of $\mathfrak{g}$. It can thus play a role similar to the Wess-Zumino consistency condition for anomalies.

  d) In general finiteness of the charges Q does not imply finiteness of the Z's. Requiring finiteness also of the Z's may impose extra conditions on the ARP's.

Example: application to Einstein Gravity

$$L = \tfrac{1}{16\pi}\sqrt{-g}(R-2\Lambda) + L_{\text{matter}}$$
$$g_{\mu\nu} = h_{\mu\nu} + \bar{g}_{\mu\nu}(x)$$

Assumption: all matter fields are negligible at $\partial\Sigma$

- The ARPs are asymptotic Killing vectors $\xi^\mu(x)$ of the background metric satisfying

$$\psi^{\mu\nu}\mathcal{L}_\xi\,\bar{g}_{\mu\nu} \longrightarrow 0.$$

- Our general formula for k_f gives in this case

$$k_\xi^{[\nu\mu]}(h,\bar{g}) = \frac{\sqrt{-\bar{g}}}{16\pi}\Big(\xi_\rho \bar{D}_\sigma H^{\rho\sigma\nu\mu} + \frac{1}{2}H^{\rho\sigma\nu\mu}\partial_\rho\xi_\sigma\Big)$$

where indices are raised and lowered with $\bar{g}_{\mu\nu}$ and

$$H^{\mu\alpha\nu\beta} = -\hat{h}^{\alpha\beta}\bar{g}^{\mu\nu} - \hat{h}^{\mu\nu}\bar{g}^{\alpha\beta} + \hat{h}^{\alpha\nu}\bar{g}^{\mu\beta} + \hat{h}^{\mu\beta}\bar{g}^{\alpha\nu}$$
$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}h_\rho{}^\rho.$$

This agrees with Abbott & Deser (1982), and reproduces for exact Killing vectors of $\bar{g}_{\mu\nu}$ the formula of Anderson & Torre (1996) and for asymptotically flat spacetimes ($\bar{g}_{\mu\nu} = \eta_{\mu\nu}$) the textbook expressions in Misner, Thorne & Wheeler and Landau & Lifshitz.

- For the possible central charges we obtain

$$\begin{aligned}
Z_{\xi',\xi} &= \frac{1}{16\pi}\int_{\partial\Sigma}(d^{n-2}x)_{\nu\mu}\sqrt{-\bar{g}}\,z^{\nu\mu} \\
z^{\nu\mu} &= -2\bar{D}_\rho\xi^\rho\,\bar{D}^\nu\xi'^\mu + 2\bar{D}_\rho\xi'^\rho\,\bar{D}^\nu\xi^\mu \\
&\quad + (\bar{D}^\rho\xi'^\nu + \bar{D}^\nu\xi'^\rho)(\bar{D}^\mu\xi_\rho + \bar{D}_\rho\xi^\mu)
\end{aligned}$$

Asymptotically AdS spacetime in 3D

- Input data (as in Brown & Henneaux 1986):

$$\bar{g}_{\mu\nu}\, dx^\mu \otimes dx^\nu = -\frac{r^2}{\ell^2}\, dt^2 + \frac{\ell^2}{r^2}\, dr^2 + r^2 d\theta^2$$

$$\Lambda = -1/\ell^2$$

$$\partial\Sigma: \quad t = t_0, \quad r \longrightarrow \infty$$

$$h_{tt} \longrightarrow O(1),\ h_{rr} \longrightarrow O(r^{-4}),\ h_{\theta\theta} \longrightarrow O(1),$$

$$h_{tr} \longrightarrow O(r^{-3}),\ h_{t\theta} \longrightarrow O(1),\ h_{r\theta} \longrightarrow O(r^{-3}).$$

- Our condition for the ARPs imposes

$$\bar{D}_t\xi_t \longrightarrow o(r^2), \quad \bar{D}_r\xi_r \longrightarrow o(r^{-2}),$$

$$\bar{D}_\theta\xi_\theta \longrightarrow o(r^2), \quad \bar{D}_t\xi_r + \bar{D}_r\xi_t \longrightarrow o(r),$$

$$\bar{D}_t\xi_\theta + \bar{D}_\theta\xi_t \longrightarrow o(r^2), \quad \bar{D}_r\xi_\theta + \bar{D}_\theta\xi_r \longrightarrow o(r).$$

These conditions are weaker than those imposed by Brown & Henneaux.
General solution in the space of functions satisfying $f \longrightarrow O(r^m) \Rightarrow \partial_r f \longrightarrow O(r^{m-1})$:

$$\begin{aligned} \xi^t &\longrightarrow \ell\, T(t,\theta), \\ \xi^r &\longrightarrow -r\partial_\theta \Phi(t,\theta) + o(r), \\ \xi^\theta &\longrightarrow \Phi(t,\theta) \end{aligned}$$

where

$$\ell\partial_t T(t,\theta) = \partial_\theta \Phi(t,\theta), \quad \ell\partial_t \Phi(t,\theta) = \partial_\theta T(t,\theta).$$

- Charges:

$$Q_\xi = \lim_{r\to\infty}\int_0^{2\pi} d\theta\, k_\xi^{[tr]}(h,\bar{g})$$

$$16\pi k_\xi^{[tr]}(h,\bar{g}) \longrightarrow -\xi^t(\frac{r^4}{\ell^4}\, h_{rr} + \frac{2}{\ell^2}\, h_{\theta\theta} - \frac{r}{\ell^2}\,\partial_r h_{\theta\theta}) \\ - \xi^\theta(2h_{t\theta} - r\partial_r h_{t\theta}).$$

These charges are finite and agree with those in Brown & Henneaux (subleading order terms of Brown & Henneaux's ξ^μ do not contribute).

- Algebra and central charges: Finiteness of central charges imposes

$$\begin{aligned} \xi^t &\longrightarrow \ell\, T(t,\theta) + O(r^{-2}), \\ \xi^r &\longrightarrow -r\partial_\theta \Phi(t,\theta) + o(r), \\ \xi^\theta &\longrightarrow \Phi(t,\theta) + O(r^{-2}). \end{aligned}$$

Still slightly weaker than in Brown & Henneaux. Central charges:

$$\begin{aligned} Z_{\xi_1,\xi_2} &= \tfrac{1}{16\pi}\lim_{r\to\infty}\int_0^{2\pi} d\theta\, \frac{2}{r}\,\partial_\theta \xi_1^r\, \partial_\theta \xi_2^t - (1 \leftrightarrow 2) \\ &= \tfrac{2\ell}{16\pi}\int_0^{2\pi} d\theta\, \partial_\theta T_1(t,\theta)\partial_\theta^2 \Phi_2(t,\theta) - (1 \leftrightarrow 2) \end{aligned}$$

Agrees with result in Brown & Henneaux and Terashima (2001). Algebra: direct product of two copies of the Virasoro algebra.

Conclusion

- Correspondence between "asymptotic reducibility parameters" (ARPs) and asymptotically conserved $n-2$ forms established. In general ARPs are not characterized by the requirement that gauge transformations with these parameters preserve the BCs for the fields, as in other approaches!

- Universal formulae for asymptotically conserved $n-2$ forms and central charges in terms of the Lagrangian and ARPs derived.

- Corresponding Lie algebras characterized, including proof of cocycle property for our expression for central charges.

- Results successfully tested (in ED, YM, GR).

- Certain ingredients of the approach can possibly be relaxed, such as the requirement that background be an exact solution of the field equations relevant near the boundary (asymptotic solution may suffice).

- Weak point: no satisfactory characterization of the general types of boundary conditions and function spaces for applicability of our methods and results – only implicit characterization through cohomological properties of d and δ; check of these properties for particular cases needed.

- Generalization to asymptotically conserved p-forms with $p < n-2$ seems straightforward (relevant to reducible gauge theories).

- Methods and results may have further applications or extensions (boundary theories, holographic principle, AdS/CFT correspondence). We are thinking about it.